Certain results on Ricci solitons in $\alpha$-Kenmotsu manifolds

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In this paper, we study some curvature problems of Ricci solitons in $\alpha$-Kenmotsu manifold. It is shown that a symmetric parallel second order-covariant tensor in a $\alpha$-Kenmotsu manifold is a constant multiple of the metric tensor. Using this result, it is shown that if $(\mathcal{L}_V g + 2S)$ is parallel where $V$ is a given vector field, then the structure $(g, V, \lambda)$ yield a Ricci soliton. Further, by virtue of this result, Ricci solitons for $n$-dimensional $\alpha$-Kenmotsu manifolds are obtained. In the last section, we discuss Ricci soliton for 3-dimensional $\alpha$-Kenmotsu manifolds.

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Introduction

A Ricci soliton are the natural generalization of Einstein metric and are defined on a Riemannian manifold. On the manifold $\mathcal{M}$, a Ricci soliton is a triple (g, $\nabla V$, $\lambda$) with a Riemannian metric $g$, a vector field $V$ and a real scalar $\lambda$ such that

$$(\mathcal{L}_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0, \quad \ldots \quad (1)$$

for any vector fields $X, Y$ on $\mathcal{M}$ where $S$ is the Ricci tensor and $\mathcal{L}_V$ denotes the Lie derivative operator along the vector field $V$. The metric satisfying (1) are very interesting in the field of physics and are often referred as quasi-Einstein. The Ricci soliton is said to be shrinking, steady and expanding according as $\lambda$ is negative, zero and positive respectively.

Das studied second order parallel tensor on an almost contact metric manifold and found that on an $\alpha$-K-contact manifold ($\alpha$ being non-zero real constant) a second order symmetric parallel tensor is a constant multiple of the associative positive definite Riemannian metric tensor. It is also proved that in an $\alpha$-Sasakian manifold there is no non-zero parallel 2-form. The study of Ricci solitons in K-contact manifolds was started by Sharma and in the continuation of this Ghosh, Sharma and Cho studied gradient Ricci soliton of a non-Sasakian $(k, \mu)$ -contact manifold. Generally, in a P-Sasakian manifold the structure vector field $\xi$ is not killing, that is $(\mathcal{L}_\xi g) \neq 0$ but in $K$-contact manifold $\xi$ is a killing vector field, that is $(\mathcal{L}_\xi g) = 0$. Recently, De$^8$ have studied Ricci soliton in P-Sasakian, Barua and De$^8$ have studied Ricci soliton in Riemannian manifolds. Since then several other studied Ricci soliton have been published in various contact manifolds and Einstein problem to Ricci soliton in f - Kenmotsu manifold, Eta-Ricci solitons on para-Kenmotsu manifolds, on contact and Lorentzian manifolds on Sasakian manifold, a -Sasakian manifold, on Kenmotsu manifold, etc.

Motivated by above studies, in this paper we treat Ricci soliton in $\alpha$-Kenmotsu manifolds. The paper is structured as follows. After
introduction, section 2 is a brief review of α-Kenmotsu manifold. Section 3, is devoted to the study of parallel symmetric second order tensor in α-Kenmotsu manifold and Ricci soliton in α-Kenmotsu manifolds. In this section, we obtain ε relation between symmetric parallel second order covariant tensor and metric tensor in α-Kenmotsu manifold. In the second problem of this section we studied the necessary and sufficient condition of a Ricci semi-symmetric α-Kenmotsu manifold and n-Einstein manifold. Section 4 is devoted to study Ricci soliton in 3-dimensional α-Kenmotsu manifold.

α-Kenmotsu manifold

An n-dimensional real $C^\omega$-manifold $M$ is said to almost contact structure $(\varphi, \xi, \eta)$ if it admits a $(1, 1)$ tensor field $\varphi$, a contravariant vector field $\xi$ and a 1-form $\eta$ which satisfy
\[
\eta(\xi) = 1, \varphi^X = -X + \eta(X)\xi, \quad \ldots \ (2)
\]
which implies
\[
\varphi(\xi) = 0, \eta(\varphi X) = 0, \quad \ldots \ (3)
\]
for all vector field $X, Y$ on $C^\omega$. Here $C^\omega$ is the Lie algebra of $C^\omega$-vector fields on $M$. An $n$-dimensional real $C^\omega$-manifold $M$ equipped with a structure $(\varphi, \xi, \eta)$ is called an almost contact manifolds.

An almost contact metric manifold $M$ with metric tensor $g$ which satisfies the conditions
\[
g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \ldots \ (4)
\]
and
\[
g(X, \xi) = \eta(X), \quad \ldots \ (5)
\]
is called an almost contact metric manifold $M$.

An almost contact metric manifold $M$ is said to be almost α-Kenmotsu manifold if
\[
da = 0, \quad \text{and} \quad d\Phi = 2a \eta \wedge \Phi,
\]
where $\Phi$ is a fundamental 2-form defined as $\Phi(X, Y) = g(\varphi X, \varphi Y)$ and $\alpha$ being a non-zero real constant. Moreover, if an almost α-Kenmotsu manifold $M$ satisfies the following relations
\[
(P_X\varphi)Y = -a(g(X, \varphi Y)\xi + \eta(Y)\varphi X), \quad \ldots \ (6)
\]
and
\[
(P_X\xi) = a(X - \eta(X)\xi), \quad \ldots \ (7)
\]
then it is called an α-Kenmotsu manifold.

On an α-Kenmotsu manifold $M$, the following relations hold.\cite{15,16,17}
\[
R(X, Y)\xi = a^2(\eta(X)Y - \eta(Y)X), \quad \ldots \ (8)
\]
\[
R(\xi, Y)\varphi = a^2(\eta(Y)X - g(X, Y)\xi), \quad \ldots \ (9)
\]
\[
\eta(R(X, Y)Z) = a^2(g(Y, X)\eta(Z) - g(Y, Z)\eta(X)), \quad \ldots \ (10)
\]
\[
S(X, \xi) = -a^2(n - 1)\eta(X), \quad \ldots \ (11)
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for all vector fields $X, Y, Z$ on $C^\omega$, where $R$ is the Ricci tensor of type $(0, 2)$ and $g$ is the Ricci operator defined as $S(X, Y) = g(QX, Y)$.

Parallel symmetric second order tensors and Ricci solitons in α-Kenmotsu manifolds

Let $h$ denote a $(0, 2)$ type symmetric tensor field which is parallel with respect to $\nabla$ that is $\nabla h = 0$. Then it follows that\cite{15,16}
\[

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replaced by $S$, that is
\[(R(X,Y) \cdot S(U,V)) = -S(R(X,Y)U,V) - S(U,R(X,Y)V). \quad \text{(21)}\]
Taking $R \cdot S = 0$ and putting $X = \xi$ in (21), we have
\[S(R(\xi,Y)U,V) + S(U,R(\xi,Y)V) = 0. \quad \text{(22)}\]
In view of (9) and $\alpha \neq 0$, the above equation becomes
\[\eta(U)S(Y,V) - g(Y,V)S(\xi,V) + \eta(V)S(U,Y) - g(Y,V)S(U,\xi) = 0. \quad \text{(23)}\]
Putting $U = \xi$ in (23) and by using (3), (11) and (12), we obtain
\[S(Y,V) = -\alpha^2(n - 1)g(Y,V).\]
This leads the following theorem.

**Theorem 2.** A Ricci semi-symmetric $\alpha$-Kenmotsu manifold is an Einstein manifold.

**Corollary 2.** If on an $\alpha$-Kenmotsu manifold the tensor field $(\mathcal{L}_g + 2S)$ is parallel, then $(g,V,\lambda)$ gives a Ricci soliton.

**Proof.** A Ricci soliton in an $\alpha$-Kenmotsu manifold is defined by (1). Thus $(\mathcal{L}_g + 2S)$ is parallel. By theorem (1) it is clear that if an $\alpha$-Kenmotsu manifold admits a symmetric parallel $(0, 2)$ tensor, then the tensor is a constant multiple of the metric tensor. Hence $(\mathcal{L}_g + 2S)$ is a constant multiple of metric tensor $g$ so that $(\mathcal{L}_g + 2S)(X,Y) = g(X,Y)h(\xi,\xi)$, where $h(\xi,\xi)$ is a non zero constant. It is the application of the theorem (1) to Ricci soliton.

**Theorem 3.** If a metric $g$ in an $\alpha$-Kenmotsu manifold is a Ricci soliton with $V = \xi$ then it is $\eta$-Einstein.

**Proof.** Putting $V = \xi$ in (1), we have
\[(\mathcal{L}_g)(X,Y) + 2S(X,Y) + 2\alpha g(X,Y) = 0, \quad \text{(24)}\]
where $(\mathcal{L}_g)(X,Y) = g(V_\xi,\xi) + g(X,V_\xi) - 2\alpha g(V_\xi,\eta). \quad \text{(25)}$
Substituting (25) in (24) and by use of (7), we obtain
\[S(X,Y) = -(\alpha + \lambda)g(X,Y) + \alpha \eta(X)\eta(Y). \quad \text{(26)}\]
Hence the result.

**Theorem 4.** A Ricci soliton $(g,\xi,\lambda)$ in an $n$-dimensional $\alpha$-Kenmotsu manifold cannot be steady but is shrinking.

**Proof.** In the Linear Algebra either the vector field $V \in \text{Span } \xi \text{ or } V \perp \xi$. However, the second case seems to be complex to analyze in practice. For this reason, we investigate for the case $V = \xi$.

By a simple computation of $(\mathcal{L}_g + 2S)$, we obtain
\[(\mathcal{L}_g)(X,Y) = 0. \quad \text{(26)}\]

\[h(\xi,\xi) = -2\lambda, \quad \text{(27)}\]
where $h(\xi,\xi) = (\mathcal{L}_g)(\xi,\xi) + 2S(\xi,\xi). \quad \text{(28)}$

Using (12) and (26) in above equation, we get
\[h(\xi,\xi) = 2\alpha^2(n - 1). \quad \text{(29)}\]
Equating (27) and (29), we have
\[\lambda = -\alpha^2(n - 1). \quad \text{(30)}\]
Since $\alpha$ is some non-zero scalar function, we have $\lambda \neq 0$, that is Ricci soliton in an $n$-dimensional $\alpha$-Kenmotsu manifold cannot be steady but is shrinking because $\lambda < 0$.

**Theorem 5.** If an $n$-dimensional $\alpha$-Kenmotsu manifold is $\eta$-Einstein then the Ricci solitons in an $\alpha$-Kenmotsu manifold that is $(g,\xi,\lambda)$ where $\lambda = -\alpha^2(n - 1)$ with varying scalar curvature cannot be steady but it is expanding.

**Proof.** The proof consists of three parts.

(i) We prove $\alpha$-Kenmotsu manifold is $\eta$-Einstein.

(ii) We prove the Ricci soliton in $\alpha$-Kenmotsu manifold is consisting of varying scalar curvature.

(iii) We find that the Ricci soliton in $\alpha$-Kenmotsu manifold is expanding.

First we prove that the $\alpha$-Kenmotsu manifold is $\eta$-Einstein: the metric $g$ is called $\eta$-Einstein if there exists two real function $a$ and $b$ such that the Ricci tensor of $g$ is given by the general equation
\[\mathcal{S}(X,Y) = ag(X,Y) + bn(X)\eta(Y). \quad \text{(30)}\]
Let $e_i (i = 1, 2, \ldots, n)$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = Y = e_i$ in (30) and taking summation over $i$, we get
\[r = an + b. \quad \text{(31)}\]
Again putting $X = Y = \xi$ in (30) then by use of (12), we have
\[a + b = -a^2(n - 1). \quad \text{(32)}\]
Then from (31) and (32), we have
\[a = \left(a^2 + \frac{r}{n - 1}\right), \quad b = \left(-a^2 + \frac{r}{n - 1}\right). \quad \text{(33)}\]
Substituting the value of $a$ and $b$ from (33) in (30), we have
\[\mathcal{S}(X,Y) = \left(a^2 + \frac{r}{n - 1}\right)g(X,Y) - \left(n a^2 + \frac{r}{n - 1}\right)\eta(X)\eta(Y). \quad \text{(34)}\]
the above equation shows that $\alpha$-Kenmotsu manifold is $\eta$-Einstein manifold.

Now, we have to show that the scalar curvature $r$ is not a constant and it is varying
For an $n$-dimensional $\alpha$-Kenmotsu manifolds the symmetric parallel covariant tensor $h(X, Y)$ of type $(0, 2)$ is given by

$$ h(X, Y) = (L_g(X, Y) + 2S(X, Y). \quad \ldots \ldots (35) $$

By using (25) and (34) in (35), we have

$$ h(X, Y) = 2\left\{\alpha(a + 1) + \frac{r}{n-1}\right\}g(X, Y) - 2\left\{\alpha(na + 1) + \frac{r}{n-1}\right\}\eta(X)\eta(Y). \quad \ldots \ldots (36) $$

Differentiating (36) covariantly with respect to $Z$ and using (14), we have

$$ (\mathcal{P}_2 h)(X, Y) = 2\left\{(Za)(\alpha(a + 1) + a(Za) + \frac{V_g}{n-1}\right\}g(X, Y) $$

$$ \quad - 2\left\{(Za)(na + 1) + n a(Za) + \frac{V_g}{n-1}\right\}\eta(X)\eta(Y) $$

$$ \quad - 2\left\{a(na + 1) + \frac{r}{n-1}\right\}a(g(Z, X) - \eta(Z)\eta(X) $$

$$ \quad + g(Z, Y) - \eta(Z)\eta(Y)). \quad \ldots \ldots (37) $$

By substituting $Z = \xi$ and $X = Y \in \text{(Span)}^\perp$ in (37) and by using $\mathcal{P}h = 0$, we have

$$ \mathcal{P}_1 = -(n - 1)\mathcal{P}_1(a(a + 1)) \quad \ldots \ldots (38) $$

On integrating (38), we have

$$ r = -(n - 1)\alpha(a + 1) + c, \quad \ldots \ldots (39) $$

where $c$ is some integral constant. Thus from (39), we have $r$ varies a scalar curvature.

Finally, we have to check the nature of the soliton that is Ricci soliton in $\alpha$-Kenmotsu manifold:

From (1), we have

$$ h(X, Y) - 2\lambda g(X, Y) \text{ then putting } X = Y = \xi, \text{ we have} $$

$$ h(\xi, \xi) = -2\lambda. \quad \ldots \ldots (40) $$

On putting $X = Y = \xi$ in (36), we have

$$ h(\xi, \xi) = -2(n - 1)\alpha^2. \quad \ldots \ldots (41) $$

Equating (40) and (41), we have

$$ \lambda = (n - 1)\alpha^2. $$

This show that $\lambda > 0, \quad \forall \ n > 1$ and hence Ricci soliton in an $\alpha$-Kenmotsu manifold is expanding.

**Theorem 6.** If a Ricci soliton $(g, \xi, \lambda)$ where $\lambda = 2\alpha^2$ of 3-dimensional $\alpha$-Kenmotsu manifold with varying scalar curvature cannot be steady but it is expanding.

**Proof.** The proof consists of three parts.

(i) We prove that the Riemannian curvature tensor of 3-dimensional $\alpha$-Kenmotsu manifold is $\eta$-Einstein.

(ii) We prove that the Ricci soliton in 3-dimensional $\alpha$-Kenmotsu manifold is consisting of varying scalar curvature.

(iii) We prove that find that the Ricci soliton in 3-dimensional $\alpha$-Kenmotsu manifold is expanding.

The Riemannian curvature tensor of 3-dimensional $\alpha$-Kenmotsu manifold is given by

$$ R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y - \frac{\lambda}{2} g(Y, Z)X - g(X, Z)Y). \quad \ldots \ldots (42) $$

Putting $Z = \xi$ in (42) and by using (8) and (11), we have

$$ a^2(\eta(X)Y - \eta(Y)X) = (\eta(Y)QX - \eta(X)QY - \frac{2}{2} g(Y, Z)X - \eta(X)Y). \quad \ldots \ldots (43) $$

Again putting $Y = \xi$ in (43) and by using (2), (3) and (13), we get

$$ QX = \left(\alpha^2 + \right) X - \left(3\alpha^2 + \right) \eta(X)\xi. \quad \ldots \ldots (44) $$

By taking an inner product with $Y$ in (44), we have

$$ S(X, Y) = \left(\alpha^2 + \right) g(X, Y) - \left(3\alpha^2 + \right) \eta(X)\eta(Y). \quad \ldots \ldots (45) $$

It shows that 3-dimensional $\alpha$-Kenmotsu manifold is $\eta$-Einstein manifold.

Now, we have to show that the scalar curvaturer is not a constant that is $r$ is varying We have

$$ h(X, Y) = (L_g(X, Y) + 2S(X, Y). \quad \ldots \ldots (46) $$

By using (25) and (45) in (46), we have

$$ h(X, Y) = 2\left\{\alpha(a + 1) + \frac{1}{n}\right\}g(X, Y) - 2\left\{\alpha(3a + 1) $$

$$ \quad + \right\}g(Y, Z)X - \eta(Y)). \quad \ldots \ldots (47) $$

Differentiating above equation with respect to $Z$, we have

$$ (\mathcal{P}_2 h)(X, Y) = 2\left\{(Za)(\alpha + 1) + a(Za) + \frac{V_g}{2}\right\}g(X, Y) $$

$$ \quad - 2\left\{(Za)(na + 1) + a(3Za) + \right\}\eta(X)\eta(Y) $$

$$ \quad - 2\left\{a(3a + 1) + \frac{1}{2}\right\}(\mathcal{P}_1 g)(X)\eta(Y) $$

$$ \quad + \eta(X)(\mathcal{P}_2 \eta(Y)). \quad \ldots \ldots (48) $$

By substituting $Z = \xi$ and $X = Y \in \text{(Span)}^\perp$ in (48) and by using $\mathcal{P}h = 0$, we have

$$ \mathcal{P}_1 = -(n - 1)\mathcal{P}_1(a(a + 1) \quad \ldots \ldots (49) $$

On integrating (49), we have

$$ r = -(n - 1)\alpha(a + 1) + c, \quad \ldots \ldots (50) $$

where $c$ is some integral constant. Thus from (50), we have $r$ is a varying scalar curvature.

Finally we have to check the nature of the Ricci soliton $(g, \xi, \lambda)$ in 3-dimensional $\alpha$-Kenmotsu manifold.

From (1), we have

$$ h(X, Y) - 2\lambda g(X, Y) \text{ then putting } X = Y = \xi, \text{ we have} $$

$$ h(\xi, \xi) = -2\lambda. \quad \ldots \ldots (51) $$

On putting $X = Y = \xi$ in (47), we have

$$ h(\xi, \xi) = -4\alpha^2. \quad \ldots \ldots (52) $$

Equating (51) and (52), we have
Equating (51) and (52), we have
\[ \lambda = 2\alpha^2. \]
This shows that \( \lambda > 0 \) and hence Ricci soliton in an \( \alpha \)-Kenmotsu manifold is expanding.

References