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On conformally flat weakly Ricci symmetric spacetimes

F. Özen Zengin

Department of Mathematics, Istanbul Technical University, Istanbul 34469, Turkey

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ABSTRACT

Tamassy and Binh introduced weakly symmetric Riemannian manifolds. The properties of weakly symmetric and weakly Ricci symmetric Riemannian manifolds are studied by some authors. In this paper, the conformally flat weakly Ricci symmetric manifolds is considered. In this case the Ricci tensor of Riemannian manifold is a quadratic Killing tensor, and some properties of this manifold are obtained. In conclusion, it is found that the energy momentum tensor of this space in a perfect fluid is a quasi-Einstein tensor and also a Codazzi tensor.

Key words: Weakly Ricci symmetric manifold; conformally flat manifold; energy-momentum tensor; quadratic Killing tensor; quadratic conformal Killing tensor; Codazzi tensor.

Mathematical subject classification: 53B15, 53B20, 53C15, 53C25

INTRODUCTION

A non-flat Riemannian manifold (M,g) can be an n-dimensional differentiable manifold of class C^∞ with metric g and the Riemannian connection ∇ (n>2). (M,g) is called weakly symmetric1 if there exists 1-forms *A, B, C, D, E.*

$$(\nabla_X R)(Y,Z,U,V) = A(X)R(Y,Z,U,V) + B(Y)R(X,Z,U,V) + C(Z)R(Y,X,U,V) + D(U)R(Y,Z,X,V) + E(V)R(Y,Z,U,X)$$

where R is the curvature tensor of (M,g) and $X, Y, Z \epsilon \chi(M)$.[1,2]

(M,g) is called weakly Ricci symmetric if the Ricci tensor S is non-zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) + B(Y)S(X,Z) + C(Z)S(Y,X) \quad (1.1)$$

where *A,B,C* are 1-forms. Such a manifold is denoted by $(WRS)_n$.

Let (M,g) be a spacetime manifold with Levi-Civita connection ∇. A quadratic Killing tensor is a generalization of a Killing vector is defined as a second order symmetric tensor T satisfying the condition

$$(\nabla_X T)(Y,Z) + (\nabla_Y T)(Z,X) + (\nabla_Z T)(X,Y) = 0 \quad (1.2)$$

Corresponding author: F. Özen Zengin
Phone. +90-2122856820 Fax. +90-2122856386
E-mail: fozen@itu.edu.tr

A quadratic conformal Killing tensor is analogous generalization of a conformal Killing vector and is defined as a second order symmetric tensor T satisfies the condition

$$(\nabla_X T)(Y,Z) + (\nabla_Y T)(Z,X) + (\nabla_Z T)(X,Y) = a(X)g(Y,Z) + a(Y)g(Z,X) + a(Z)g(X,Y) \quad (1.3)$$

for a smooth 1-form $a(X)$ on M. The above equation is equivalent $T(l,l)$ that be constant along null geodesics with parallely propagated tangent vector l.[3,4,5] It is shown that[4] how the fourth first integral of the geodesic equation arises from the existence of a quadratic Killing tensor. Moreover, they proved that every vacuum solution(with or without cosmological constant) of Einstein's equations, whose Weyl tensor is of type {2,2}admits a quadratic conformal Killing tensor that is reducible (i.e. cannot be constructed as a linear combination of symmetrized tensor products of conformal Killing vectors) provided the spacetime admits two and only two independent Killing vectors. It is shown that[5] the quadratic Killing tensor can be used to get Carter's fourth first integral of the equation of motion for charged test particles. They further showed that the energy-momentum tensor of certain test electromagnetic field is a quadratic conformal Killing tensor and, in particular, showed that the charged Kerr spacetime admits a quadratic Killing tensor. Sharma and Ghosh[6] showed that the energy-momentum tensor T of an expanding perfect fluid spacetime (M,g) is a nontrivial (i.e. non-Killing) conformal Killing if and only if M is shear-free, vorticity-free and satisfies certain differential conditions on the energy-density and pressure in terms of the expansion and acceleration. They also showed that when the energy-momentum tensor is Killing, M is expansion-free and shear-free and its flow is geodesic (not necessarily vorticity-free), and furthermore that its energy-density and pressure are constant over M.

In this paper, the conditions that the Ricci tensor of a weakly Ricci symmetric Riemannian manifold be a quadratic Killing tensor or a quadratic conformal Killing tensor are obtained. After that, assuming that our manifold is conformally flat spacetime, then it is proved that the energy-momentum tensor satisfying the Einstein's equations with a cosmological constant of this manifold is an Einstein tensor. It is also shown that this energy-momentum tensor is a Codazzi tensor.

CONFORMALLY FLAT $(WRS)_N$ SPACE-TIMES

Let us consider that M is a weakly Ricci symmetric Riemannian manifold. In this case, we denote M by $(WRS)_n$ (n<2). In conformally flat $(WRS)_n$ with definite metric if $A(X) \neq 0$ then the scalar curvature r is non-zero and the Ricci tensor is of the form.[7]

$$S(X,Y) = rv(X)v(Y) \quad (2.1)$$

In addition, it is proved that[8] the scalar curvature of a conformally flat weakly Ricci symmetric space is constant. The curvature tensor of a conformally flat $(WRS)_n$ is

$$\begin{aligned} R(X,Y,Z,W) = {} & a\big(g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\big) \\ & + b\big(g(X,Z)v(Y)v(W) - g(X,W)v(Y)v(Z) \\ & - g(Y,Z)v(X)v(W) + b\big(g(X,Z)v(Y)v(W) \\ & - g(X,W)v(Y)v(Z) - g(Y,Z)v(X)v(W) \\ & + g(Y,W)v(X)v(Z)\big) \end{aligned} \quad (2.2)$$

where

$$a = -\frac{r}{(n-1)(n-2)}, \qquad b = -\frac{r}{n-2}$$

and r is the scalar curvature of our manifold.

Now, we can state the following theorems:

THEOREM 2.1.

In a $(WRS)_n$, if the Ricci tensor is a quadratic Killing tensor then the sum of the associated 1-forms must be zero ($n>3$).

Proof. From (1.1), we can get

$$(\nabla_Z S)(X,Y) = A(Z)S(X,Y) + B(X)S(Z,Y) + C(Y)S(X,Z) \quad (2.3)$$

and $(\nabla_Y S)(Z,X) = A(Y)S(Z,X) + B(Z)S(Y,X) + C(X)S(Z,Y)$ (2.4)

Adding (1.1), (2.3) and (2.4) side by side, we find

$$(\nabla_Z S)(X,Y) + (\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) = \big(A(Z)+B(Z)+C(Z)\big)S(X,Y) + \big(A(X)+B(X)+C(X)\big)S(Y,Z) + \big(A(Y)+B(Y)+C(Y)\big)S(X,Z) \quad (2.5)$$

If $S(X,Y)$ is a quadratic Killing tensor, from (1.2) and (2.5), it can be obtained that

$$\alpha(Z)S(X,Y) + \alpha(X)S(Y,Z) + \alpha(Y)S(X,Z) = 0 \quad (2.6)$$

In Walker's Lemma,[9] it is said that if $a(X,Y)$ and $b(X)$ are the numbers satisfying $a(X,Y) = a(Y,X)$ and

$$a(X,Y)b(Z) + a(Y,Z)b(X) + a(Z,X)b(Y) = 0 \quad (2.7)$$

for all X, Y, Z. Then either $a(X,Y)$ are all zero or all the $b(X)$ are zero. Hence, by the above Lemma, we get from (2.6) and (2.7), either $\alpha(Z) = 0$ or $S(X,Y) = 0$. By the definition of $(\mathrm{WRS})_n$, $S(X,Y) \neq 0$. In this case, $\alpha(Z) = A(Z) + B(Z) + B(Z) = 0$. The proof is completed.

THEOREM 2.2.

In a conformally flat $(\mathrm{WRS})_n$, the Ricci tensor of this manifold is Codazzi tensor.

Proof. The Cotton tensor[10] is defined in terms of the Ricci tensor $S(X,Y)$ and the scalar curvature r of the metric as in the following form

$$\mathrm{C}(\mathrm{X},\mathrm{Y},\mathrm{Z}) = (\nabla_Z S)(X,Y) - (\nabla_Y S)(X,Z) + \frac{1}{2(n-1)}\big((\nabla_{\mathrm{Y}}\mathrm{r})g(X,Z) - (\nabla_{\mathrm{Z}}\mathrm{r})g(X,Y)\big) \quad (2.8)$$

A metric is conformally flat if and only if its Cotton tensor is identically zero.[11] And we know that in this manifold, the scalar curvature is constant. Thus, (2.8) reduces to

$$(\nabla_Z S)(X,Y) - (\nabla_Y S)(X,Z) = 0 \quad (2.9)$$

From (2.9), the proof is clear.

THEOREM 2.3.

A conformally flat $(\mathrm{WRS})_n$ can not admit a quadratic conformal Killing tensor.

Proof. In a conformally flat $(\mathrm{WRS})_n$, we have the form (2.1).

Taking the covariant derivative of (2.1) and remembering that the scalar curvature of this manifold is constant, we find

$$(\nabla_X S)(Y,Z) = r\big((\nabla_X v)(Y)v(Z) + (\nabla_X v)(Z)v(Y)\big) \quad (2.10)$$

By putting (2.10) and (2.9) in (1.3), we obtain

$$3r\big((\nabla_Y v)(X)v(Z) + (\nabla_Y v)(Z)v(X)\big) = a(Z)g(X,Y) + a(Y)g(X,Z) + a(X)g(Y,Z) \quad (2.11)$$

From (2.10) and (2.11), it can be seen the

$$3(\nabla_Z S)(X,Y) = a(Z)g(X,Y) + a(Y)g(X,Z) + a(X)g(Y,Z) \quad (2.12)$$

If we take $X = Y$ in (2.12), we can get

$$3(\nabla_Z r) = (n+2)a(Z) \quad (2.13)$$

Remembering that the scalar curvature of this manifold is constant, (2.13) reduces to

$a(Z) = 0$

Thus, we can say that a conformally flat $(\mathrm{WRS})_n$ can not admit a quadratic conformal Killing Ricci tensor.

Now, we consider that our space is a perfect fluid. A perfect fluid is a spacetime (M,g) satisfying the Einstein's equations

$$\mathrm{S}(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = kT(X,Y) \quad (2.14)$$

where $S(X,Y)$ and r denote the Ricci tensor and the scalar curvature, respectively. λ is the cosmological constant and $T(X,Y)$ is the energy-momentum tensor.

We can state the following theorem:

THEOREM 2.4.

For a perfect fluid in a conformally flat $(WRS)_4$, the energy-momentum tensor satisfying the Einstein's equations with a cosmological constant is a quasi-Einstein tensor.

Proof. In a conformally flat $(WRS)_4$, from (2.14) and (2.1), we find

$$T(X,Y)=\frac{1}{k}\left(\lambda-\frac{1}{2}r\right)g(X,Y)+\ \frac{r}{k}v(X)v(Y)$$

Thus, we find that

$$T(X,Y)=ag(X,Y)+bv(X)v(Y) \qquad (2.15)$$

where $a=\frac{1}{k}\left(\lambda-\frac{1}{2}r\right)$ and $b=\frac{r}{k}$

THEOREM 2.5.

For a perfect fluid in a conformally flat $(WRS)_4$, the energy-momentum tensor satisfying the Einstein's equations with a cosmological constant is a Codazzi tensor.

Proof. By taking the covariant derivative of (2.15) and remembering that the scalar curvature of this space is constant, we obtain that

$$(\nabla_Z T)(X,Y)=\frac{1}{k}(\nabla_Z S)(X,Y) \qquad (2.16)$$

By changing the indices in (2.16), it can be found that

$$(\nabla_Y T)(X,Z)=\frac{1}{k}(\nabla_Y S)(X,Z) \qquad (2.17)$$

Subtracting (2.17) from (2.16), we get

$$(\nabla_Z T)(X,Y)-(\nabla_Y T)(X,Z)=\frac{1}{k}((\nabla_Z S)(X,Y)-(\nabla_Y S)(X,Z)) \qquad (2.18)$$

By using Theorem 2.2 and (2.18), we can easily seen that

$$(\nabla_Z T)(X,Y)=(\nabla_Y T)(X,Z)$$

Thus, the proof is completed.

REFERENCES

1. Tamassy L & Binh TQ (1989). On weakly symmetric and weakly projective symmetric Riemannian manifolds. *Coll Math Soc J Bolyai*, **56**, 663-670.
2. Tamassy L & Binh TQ (1993). On weakly symmetrics of Einstein and Sasakian manifolds. *Tensor (N.S.)*, **53**, 140-148.
3. Wald RM (1984). *General Relativity*. The University of Chicago Press, Chicago.
4. Walker M & Penrose R (1970). On quadratic first integrals of geodesic equations for {22} spacetimes. *Commun Math Phys*, **18**, 265.
5. Hughston LP, Penrose R, Sommers P & Walker M (1972). On a quadratic first integral for the charged particle orbits in the charged Kerr solution. *Commun Math Phys*, **27**, 303.
6. Sharma R & Ghosh A (2010). Perfect fluid space-times whose energy-momentum tensor is conformal Killing. *J Math Phys*, **51**, 1.
7. De UC & Ghosk SK (2000). On conformally flat pseudosymmetric spaces. *Balkan J Geom Appl*, **5**, 61-64.
8. De UC & Ghosk SK (2002). On weakly Ricci symmetric spaces. *Publ Math Debrecen*, **60**,17.
9. Walker AG (1950). On Ruse's spaces of recurrent curvature. *Proc Lond Math Soc*, **52**, 36-54.
10. Schouten JA (1921). Uber die konforme Abbildung n-dimensionaler Mannigfaltigkeiten mit quadratischer Maßbestßimmung auf eine Mannigfaltigkeit mit euklidischer Maßbesttimmung. *Math Zeitschrift*, **11**, 58-88

