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Stress-strain in anisotropic solid medium

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ABSTRACT

The numbers of elastic constants in the anisotropic medium has been investigated through the Cartesian co-ordinate plane symmetry. We have observed different types of anisotropic medium have different number of elastic constants. These constants are related with the properties of the medium.

Key words: Anisotropic medium; monoclinic medium; orthotropic medium; transversely isotropic medium; isotropic medium.

INTRODUCTION

A uniform material which contains an internal structure (such as crystal), so that elastic properties vary with direction, is defined as anisotropic elastic medium. The variation of properties for purely elastic solids containing crystals can be fully described by a fourth-order tensor of anisotropic elastic constants. Theoretical seismology aims to study earthquake phenomenon with the application of mathematical methods. Earthquakes and other disturbances generate seismic waves, which give information about both the source and the material they pass through. An earthquake is the result of release of energy somewhere inside the earth. This released energy sets up various types of elastic waves which are transmitted in the earth with definite velocities depending on the density and the elastic parameters of the materials in the earth.[1]

Seismologists observed that the seismic waves propagated by sedimentary rocks do not always behave as if the medium were isotropic. The property of velocity anisotropy, hereafter referred to, simply as anisotropy, is characterized by the variations of velocity with the direction of propagation. Many researchers investigated the problem of anisotropic elastic medium. Song and Helmberger[2] investigated the problem of anisotropy of Earth's inner core. They explained the different forms of anisotropy in the earth's inner core. Musgrave[3] showed the theoretically and numerically that there are fundamental differences between the propagation of seismic waves in isotropic and anisotropic media. These differences are subtle and difficult to detect on observed seismograms. Crampin[4] observed that the calculation of synthetic seismograms is an important technique for deter-

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mining the effects of anisotropy on seismic wave propagation in any particular earth's structure. Singh[5] studied the quasi nature of elastic waves in the anisotropic medium. He had shown that the quasi nature of elastic waves is due to the phase velocity of elastic waves in the anisotropic medium depending on the angle of propagation of the medium. Auld[6] studied the problems of acoustic fields and waves in solids. Malvern[7] attempted the problem on the introduction to the mechanics of a continuous medium.

In this work, the fundamental of anisotropic medium has been discussed. We have observed that different types of anisotropic medium have different number of elastic constants to represent the stress–strain relationship. We study the generalized anisotropic medium, monoclinic medium, orthotropic medium, hexagonal symmetry, transversely isotropic medium, cubic medium and isotropic medium which are having different number of elastic constants.

HOOKE'S LAW

The generalized Hooke's Law says that for sufficiently small strains, each component of stress tensor is a linear combination of the components of strain tensor. The coefficients in the linear form connecting the components of these tensors are elastic constants. In the generalized Hooke's Law, there are thirty six elastic constants and is given as

$$\mathfrak{I}_{ij} = C_{ijkl}e_{kl},\ (i, j, k, l = 1, 2, 3);\ C_{ijkl} = C_{jikl} = C_{ijlk} \qquad 2.1$$

Let us introduced the following notations to avoid dealing with double sums

$$\mathfrak{I}_1 = \mathfrak{I}_{11} \quad \mathfrak{I}_2 = \mathfrak{I}_{22} \quad \mathfrak{I}_3 = \mathfrak{I}_{33}$$
$$\mathfrak{I}_4 = \mathfrak{I}_{23} \quad \mathfrak{I}_5 = \mathfrak{I}_{31} \quad \mathfrak{I}_6 = \mathfrak{I}_{12} \qquad 2.2$$
$$e_1 = e_{11} \quad e_2 = e_{22} \quad e_3 = e_{33}$$
$$e_4 = 2e_{23} \quad e_5 = 2e_{31} \quad e_6 = 2e_{12} \qquad 2.3$$

Equation (2.1) may be re-written as

$$\mathfrak{I}_i = C_{ij}e_j\ (i, j = 1, 2, .., 6) \qquad 2.4$$

which may be represented the following matrix as

$$\begin{bmatrix}\mathfrak{I}_1\\ \mathfrak{I}_2\\ \mathfrak{I}_3\\ \mathfrak{I}_4\\ \mathfrak{I}_5\\ \mathfrak{I}_6\end{bmatrix} = \begin{bmatrix}C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36}\\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46}\\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56}\\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}\end{bmatrix}\begin{bmatrix}e_1\\ e_2\\ e_3\\ e_4\\ e_5\\ e_6\end{bmatrix} \qquad 2.5$$

Or in terms of co-ordinates

$$\begin{bmatrix}\mathfrak{I}_{xx}\\ \mathfrak{I}_{yy}\\ \mathfrak{I}_{zz}\\ \mathfrak{I}_{yz}\\ \mathfrak{I}_{zx}\\ \mathfrak{I}_{xy}\end{bmatrix} = \begin{bmatrix}C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36}\\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46}\\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56}\\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}\end{bmatrix}\begin{bmatrix}e_{xx}\\ e_{yy}\\ e_{zz}\\ 2e_{yz}\\ 2e_{zx}\\ 2e_{xy}\end{bmatrix}$$

where
$$\mathfrak{I}_{xx} = \mathfrak{I}_1 \quad \mathfrak{I}_{yy} = \mathfrak{I}_2 \quad \mathfrak{I}_{zz} = \mathfrak{I}_3$$
$$\mathfrak{I}_{yz} = \mathfrak{I}_4 \quad \mathfrak{I}_{zx} = \mathfrak{I}_5 \quad \mathfrak{I}_{xy} = \mathfrak{I}_6$$
$$e_{xx} = e_1 \quad e_{yy} = e_2 \quad e_{zz} = e_3$$
$$2e_{yz} = e_4 \quad 2e_{zx} = e_5 \quad 2e_{xy} = e_6$$

The maximum number of elastic constants is thirty six in this case.

The strain energy density function (W) of the anisotropic medium is defined by

$$W = \frac{1}{2} C_{ij}e_ie_j \qquad 2.6$$

with the property that

$$\frac{\partial W}{\partial e_i} = \mathfrak{I}_i\ ; \qquad (i = 1, 2, 3) \qquad 2.7$$

Differentiating Equation (2.4) with respect to e_k, we get

$$\frac{\partial W}{\partial e_k} = \frac{1}{2} Cij \left(\frac{\partial e_i}{\partial e_k} e_j + e_i \frac{\partial e_j}{\partial e_k}\right)$$
$$= \frac{1}{2} C_{ij} (\delta_{ik}\square\ e_j + e_i\delta_{jk})$$
$$= \frac{1}{2} C_{ki}\, e_i + \frac{1}{2} C_{ik}\, e_i \qquad 2.8$$

(Because j is dummy suffix)

By Equations (2.4) and (2.7), we have

$$\mathfrak{I}_k = C_{ki}e_i$$

and $\frac{\partial W}{\partial ek} = C_{ki}e_i$

With the help of Equation (2.8)

$\frac{1}{2} C_{ki}\, e_j + \frac{1}{2} C_{ik}\, e_i = C_{ki}e_i$, which implies that

$C_{ki} = C_{ik}$ or $C_{ij} = C_{ji}$ (i, j = 1, 2, 3), i.e., C_{ij}

is symmetric.

This makes 21 elastic constants. Thus twenty-one independent elastic constants form a general anisotropic body. If any medium is elastically symmetric in certain direction, then the number of independent constants may be less than 21.

STRESS AND STRAIN UNDER THE TRANSFORMED OF AXIS

Let us transform the coordinate axis from (X, Y, Z) to (X′ , Y′ , Z′) with the following direction cosines given by

	X	Y	Z
X′	l_{11}	l_{12}	l_{13}
Y′	l_{21}	l_{22}	l_{23}
Z′	l_{31}	l_{32}	l_{33}

In the new transformed coordinate system, the generalized Hooke's Law is given as

$\Im'_i ; = C_{ij} e'_j$ (i, j = 1, 2, …., 6) 3.1

The new stress traction may be written as

$\Im'_{\alpha\beta} = l_{\alpha i} l_{\beta j} \Im_{ij}$; (α, β, i, j = 1, 2, 3) 3.2

Thus $\Im'_{11} = l_{1i} l_{1j} \Im_{ij}$; (i, j = 1, 2, 3)

$= l_{11}l_{11}\Im_{11}+l_{11}l_{12}\Im_{12}+l_{11}l_{13}\Im_{13}+l_{12}l_{11}\Im_{21}+l_{12}l_{12}\Im_{22}+l_{12}l_{13}\Im_{23}+l_{13}l_{11}\Im_{31}+l_{13}l_{12}\Im_{32}+l_{13}l_{13}\Im_{33}$ 3.2a

$\Im'_{22} = l_{2i}l_{2j}\Im_{ij}$; (i, j = 1, 2, 3)

$= l_{21}l_{21}\Im_{11}+l_{21}l_{22}\Im_{12}+l_{21}l_{23}\Im_{13}+l_{22}l_{21}\Im_{21}+l_{22}l_{22}\Im_{22}+l_{22}l_{23}\Im_{23}+l_{23}l_{21}\Im_{31}+l_{23}l_{22}\Im_{32}+l_{23}l_{23}\Im_{33}$ 3.2b

$\Im'_{33} = l_{3i}l_{3j}\Im_{ij}$; (i, j = 1, 2, 3)

$= l_{31}l_{31}\Im_{11}+l_{31}l_{32}\Im_{12}+l_{32}l_{31}\Im_{21}+l_{32}l_{32}\Im_{22}+l_{32}l_{32}\Im_{22}+l_{32}l_{33}\Im_{23}+l_{33}l_{31}\Im_{31}+l_{33}l_{32}\Im_{32}+l_{33}l_{33}\Im_{33}$ 3.2c

$\Im'_{23} = l_{2i}l_{3j}\Im_{ij}$; (i, j = 1, 2, 3)

$= l_{21}l_{31}\Im_{11}+l_{21}l_{32}\Im_{12}+l_{21}l_{33}\Im_{13}+l_{22}l_{31}\Im_{21}+l_{22}l_{32}\Im_{22}+l_{22}l_{33}\Im_{23}+l_{23}l_{31}\Im_{31}+l_{23}l_{32}\Im_{32}+l_{23}l_{33}\Im_{33}$ 3.2d

$\Im'_{31} = l_{3i}l_{1j}\Im_{ij}$; (i, j = 1, 2, 3)

$= l_{31}l_{11}\Im_{11}+l_{31}l_{12}\Im_{12}+l_{31}l_{13}\Im_{13}+l_{32}l_{11}\Im_{21}+l_{32}l_{12}\Im_{22}+l_{32}l_{13}\Im_{23}+l_{33}l_{11}\Im_{31}+l_{33}l_{12}\Im_{32}+l_{33}l_{13}\Im_{33}$ 3.2e

$\Im'_{12} = l_{1i}l_{2j}\Im_{ij}$; (i, j = 1, 2, 3)

$= l_{11}l_{21}\Im_{11}+l_{11}l_{22}\Im_{12}+l_{11}l_{23}\Im_{13}+l_{12}l_{21}\Im_{21}+l_{12}l_{22}\Im_{22}+l_{12}l_{23}\Im_{23}+l_{13}l_{21}\Im_{31}+l_{13}l_{12}\Im_{32}+l_{13}l_{13}\Im_{33}$ 3.2f

The new strain components may be written as

$e'_{\alpha\beta} = l_{\alpha i} l_{\beta j} e_{ij}$; (α, β, i, j = 1, 2, 3) 3.3

Thus $e'_{11} = l_{11}l_{11}e_{11}+l_{11}l_{12}e_{12}+l_{11}l_{13}e_{13}+l_{12}l_{11}e_{21}+l_{12}l_{12}e_{22}+l_{12}l_{13}e_{23}+l_{13}l_{11}e_{31}+l_{13}l_{12}e_{32}+l_{13}l_{13}e_{33}$ 3.3a

$e'_{22} = l_{21}l_{21}e_{11}+l_{21}l_{22}e_{12}+l_{21}l_{23}e_{13}+l_{22}l_{21}e_{21}+l_{22}l_{22}e_{22}+l_{22}l_{23}e_{23}+l_{23}l_{21}e_{31}+l_{23}l_{22}e_{32}+l_{23}l_{23}e_{33}$ 3.3b

$e'_{33} = l_{31}l_{31}e_{11}+l_{31}l_{32}e_{12}+l_{32}l_{31}e_{21}+l_{32}l_{32}e_{22}+l_{32}I_{32}e_{22}+l_{32}l_{33}e_{23}+l_{33}l_{31}e_{31}+l_{33}l_{32}e_{32}+l_{33}l_{33}e_{33}$ 3.3c

$e'_{23} = l_{21}l_{31}e_{11}+l_{21}l_{32}e_{12}+l_{21}l_{33}e_{13}+l_{22}l_{31}e_{21}+l_{22}l_{31}e_{21}+l_{22}l_{32}e_{22}+l_{22}l_{33}e_{23}+l_{23}l_{32}e_{32}+l_{23}l_{33}e_{33}$ 3.3d

$e'_{31} = l_{31}l_{11}e_{11}+l_{31}l_{12}e_{12}+l_{31}l_{13}e_{13}+l_{32}l_{11}e_{21}+l_{32}l_{12}e_{22}+l_{32}l_{13}e_{23}+l_{33}l_{11}e_{31}+l_{33}l_{12}e_{32}+l_{33}l_{13}e_{33}$ 3.3e

$e'_{12} = l_{11}l_{21}\Im_{11}+l_{11}l_{22}e_{12}+l_{11}l_{23}e_{13}+l_{12}l_{21}e_{21}+l_{12}l_{22}e_{22}+l_{12}l_{23}e_{23}+l_{13}l_{21}e_{31}+l_{13}l_{12}e_{32}+l_{13}l_{13}e_{33}$ 3.3f

Thus, we have seen that there are different values of stress tensors and strain tensors for the new co-ordinate systems. This makes us possible to find the different numbers of elastic constants for different elastic plane symmetry.

MEDIUM WITH ONE PLANE OF ELASTIC SYMMETRY

A substance is considered elastically symmetric with respect to XY-plane (or Z-plane). Under this symmetry, the elastic constants are invariant in the transformation

X = X′, Y and *Z = -Z′*

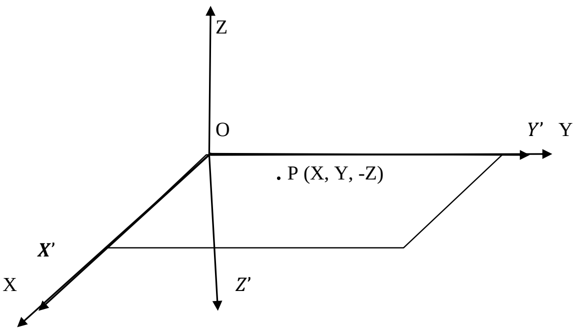

Figure 1. Reflection symmetry across XY-plane.

The direction cosines for this transformation are given as

	X	Y	Z
X′	1	0	0
Y′	0	1	0
Z′	0	0	-1

With the help of Equations (3.2a-f) and (3.3a-f), we have

$\Im'_1 = \Im_1 \quad \Im'_2 = \Im_2 \quad \Im'_3 = \Im_3$

$\Im'_4 = -\Im_4 \quad \Im'_5 = -\Im_5 \quad \Im'_6 = \Im_6 \qquad 4.1$

and $e'_1 = e_1 \quad e'_2 = e_2 \quad e'_3 = e_3$

$e'_4 = -e_4 \quad e'_5 = -e_5 \quad e'_6 = e_6 \qquad 4.2$

Using Equations (4), (10), (13) and (14), we get

$\Im_1 = \Im'_1$

or $C_{11}e_1 + C_{12}e_2 + C_{13}e_3 + C_{14}e_4 + C_{15}e_5 + C_{16}e_6$

$= C_{11}e'_1 + C_{12}e'_2 + C_{13}e'_3 + C_{14}e'_4 + C_{15}e'_5 + C_{16}e'_6$

or $C_{11}e_1 + C_{12}e_2 + C_{13}e_3 + C_{14}e_4 + C_{15}e_5 + C_{16}e_6$

$= C_{11}e_1 + C_{12}e_2 + C_{13}e_3 - C_{14}e_4 - C_{15}e_5 + C_{16}e_6$

or $C_{14}e_4 + C_{15}e_5 = -C_{14}e_4 - C_{15}e_5$

Comparing the coefficients of e_4 and e_5, we have

$C_{14} = -C_{14}$ and $C_{15} = -C_{15}$ which implies $C_{14} = 0$ and $C_{15} = 0$.

Similarly, $\Im'_2 = \Im_2$ gives $C_{24} = 0$ and $C_{25} = 0$;

$\Im'_3 = \Im_3$ gives $C_{34} = 0$ and $C_{35} = 0$;

$\Im'_4 = -\Im_4$ gives $C_{41} = C_{42} = C_{43} = C_{46} = 0$;

$\Im'_5 = -\Im_5$ gives $C_{51} = C_{52} = C_{53} = C_{56} = 0$;

$\Im'_6 = \Im_6$ gives $C_{64} = C_{65} = 0$.

The system of Equation (2.5) may be rewritten as

$$\begin{bmatrix}\Im_1\\ \Im_2\\ \Im_3\\ \Im_4\\ \Im_5\\ \Im_6\end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16}\\ C_{21} & C_{22} & C_{23} & 0 & 0 & C_{26}\\ C_{31} & C_{32} & C_{33} & 0 & 0 & C_{36}\\ 0 & 0 & 0 & C_{44} & C_{45} & 0\\ 0 & 0 & 0 & C_{54} & C_{55} & 0\\ C_{61} & C_{62} & C_{63} & 0 & 0 & C_{66}\end{bmatrix}\begin{bmatrix}e_1\\ e_2\\ e_3\\ e_4\\ e_5\\ e_6\end{bmatrix}$$

4.3

There are only thirteen elastic constants. A medium in which the elastic system is represented by thirteen elastic constants is known as anisotropic medium with monoclinic symmetry or monoclinic medium.

Medium with two plane of elastic symmetry

A medium of two plane elastic symmetry is generally denoted as orthotropic medium. In such a case, we choose the axis of the co-ordinates so that the co-ordinate planes coincide with the planes of elastic symmetry as shown in Figure 2.

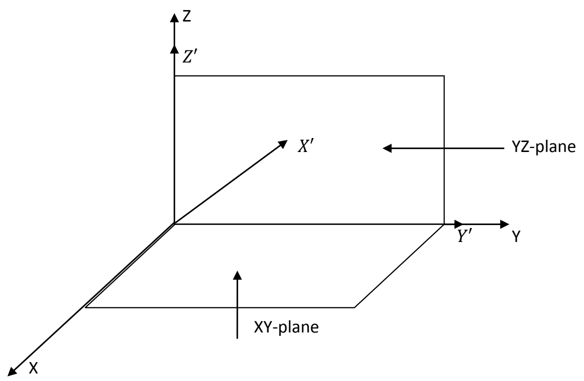

Figure 2. Reflection symmetry across two mutually orthogonal planes (XY & YZ).

Under this condition, a few elastic constants in Equation (4.3) vanish. The elastic constants C_{ij} are symmetric with respect to both XY-plane & YZ-plane. The stress-strain relation is invariant under this transformation of co-ordinates and the direction cosines are as follows

	X	Y	Z
X'	-1	0	0
Y'	0	1	0
Z'	0	0	1

$X = -X$; $Y' = Y$; $Z' = Z$

By the Equations (3.2a-f) and (3.3a-f), we have

$\Im'_1 = \Im_1$ $\Im'_2 = \Im_2$ $\Im'_3 = \Im_3$

$\Im'_4 = -\Im_4$ $\Im'_5 = -\Im_5$ $\Im'_6 = \Im_6$ 5.1

and $e'_1 = e_1$ $e'_2 = e_2$ $e'_3 = e_3$

$e'_4 = -e_4$ $e'_5 = -e_5$ $e'_6 = e_6$ 5.2

Now by $\Im'_1 = \Im_1$

$C_{11}e_1 + C_{12}e_2 + C_{13}e_3 + C_{16}e_6$

$= C_{11}e'_1 + C_{12}e'_2 + C_{13}e'_3 + + C_{16}e'_6$

or $C_{11}e_1 + C_{12}e_2 + C_{13}e_3 + C_{16}e_6$

$= C_{11}e_1 + C_{12}e_2 + C_{13}e_3 - C_{16}e_6$

or $C_{16}e_6 = -C_{16}e_6$ or $C_{16} = -C_{16}$

or $C_{16} = 0$

Similarly

$\Im'_2 = \Im_2$ gives $C_{26} = 0$

$\Im'_3 = \Im_3$ gives $C_{36} = 0$

$\Im'_4 = \Im_4$ gives $C_{45} = 0$

$\Im'_5 = -\Im_5$ gives $C_{54} = 0$

$\Im'_6 = -\Im_6$ gives $C_{61} = C_{62} = C_{63} = 0$

and the system given in Equation (4.3) reduces to

$$\begin{bmatrix}\Im_1\\ \Im_2\\ \Im_3\\ \Im_4\\ \Im_5\\ \Im_6\end{bmatrix} = \begin{bmatrix}C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0\\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & C_{66}\end{bmatrix}\begin{bmatrix}e_1\\ e_2\\ e_3\\ e_4\\ e_5\\ e_6\end{bmatrix}$$

5.3

There are only nine independent elastic constants. A medium in which the elastic system is represented by these nine constants is known as anisotropic medium with orthorhombic symmetry or orthotropic medium.

ROTATION OF AXIS

Rotation about X-axis

In this case, the co-ordinate axis is rotated about X axis, i.e. X-axis kept fixed and rotated the system in such a way that Y axis equal to Z axis and Z-axis equal to negative of Y axis.

The direction cosines for this transformation are given as

	X	Y	Z
X'	1	0	0
Y'	0	0	1
Z'	0	-1	0

By the Equations (3.2a-f) and (3.3a-f), we have

$\Im'_1 = \Im_1$ $\Im'_2 = \Im_2$ $\Im'_3 = \Im_3$

$\Im'_4 = -\Im_4$ $\Im'_5 = -\Im_6$ $\Im'_6 = \Im_5$ 6.1

and $e'_1 = e_1$ $e'_2 = e_2$ $e'_3 = e_3$

$e'_4 = -e_4$ $e'_5 = -e_6$ $e'_6 = e_5$ 6.2

Now, by $\Im'_1 = \Im_1$, we have

$C_{11}e_1 + C_{12}e_2 + C_{13}e_3 = C_{11}e'_1 + C_{12}e'_2 + C_{13}e'_3$

or $C_{11}e_1 + C_{12}e_2 + C_{13}e_3 = C_{11}e_1 + C_{12}e_3 + C_{13}e_2$

or $C_{12}e_2 + C_{13}e_3 = C_{13}e_2 + C_{12}e_3$

On comparing the coefficients of e_2 & e_3

$C_{12} = C_{13}$

Similarly, by $\Im'_2 = \Im_2$ gives $C_{33} = C_{22}$ and $\Im'_5 = -\Im_6$ gives $C_{66} = C_{55}$

Hence the system of elastic constants in Equation (5.3) reduces to

$$\begin{bmatrix} \Im_1 \\ \Im_2 \\ \Im_3 \\ \Im_4 \\ \Im_5 \\ \Im_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{bmatrix} \quad 6.3$$

Under this system, the number of elastic constants is reduced to six. A medium in which the elastic system is represented by six constants is known as anisotropic medium with hexagonal symmetry.

Rotation about Y-axis

The direction cosines for this transformation are given by

	X	Y	Z
X′	0	-1	0
Y′	-1	0	0
Z′	0	0	1

By the Equations (3.2a-f) and (3.3a-f), we have

$\Im'_1 = \Im_2$ $\quad \Im'_2 = \Im_1$ $\quad \Im'_3 = \Im_3$

$\Im'_4 = -\Im_5$ $\quad \Im'_5 = -\Im_4$ $\quad \Im'_6 = \Im_5$ $\quad$ 5.1

and $e'_1 = e_2$ $\quad e'_2 = e_1$ $\quad e'_3 = e_3$

$e'_4 = -e_5$ $\quad e'_5 = -e_4$ $\quad e'_6 = e_5$ $\quad$ 5.2

By using Equations (6.4) and (6.5)

$\Im'_1 = \Im_2$

or $C_{11}e'_1 + C_{12}e'_2 + C_{12}e'_3 = C_{12}e_1 + C_{22}e_2 + C_{23}e_3$

Or $C_{11}e_2 + C_{12}e_1 + C_{12}e_3 = C_{12}e_1 + C_{22}e_2 + C_{23}e_3$

Comparing the coefficients of e_2 and e_3

$C_{11} = C_{22}$ and $C_{12} = C_{23}$

Similarly by $\Im'_4 = -\Im_5$ gives $C_{44} = C_{66}$.

The system of Equation (6.3) changes to

$$\begin{bmatrix} \Im_1 \\ \Im_2 \\ \Im_3 \\ \Im_4 \\ \Im_5 \\ \Im_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{bmatrix} \quad 6.6$$

There are only three (C_{11}, C_{12}, C_{66}) independent elastic constants which describe the elastic property of the medium. A medium in which the elastic property is represented by only three constants is known as anisotropic medium with cubic symmetry or cubic medium.

Rotation about Z-axis

Keeping Z-axis fixed and rotating the co-ordinate system through an angle Π/4 to give a new co-ordinate system (X' , Y' , Z') as

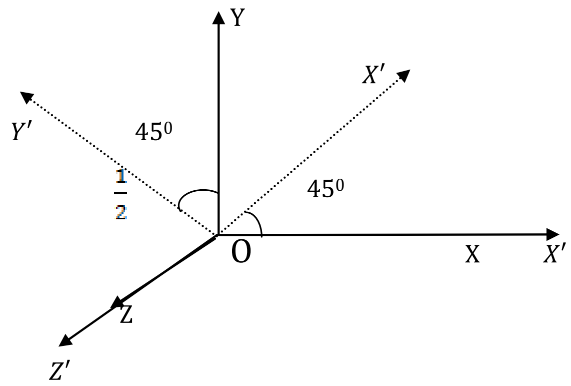

Figure 3. Rotation of XY-plane around Z-axis by Π/4.

The direction cosines relating the Z-systems are given by

	X	Y	Z
X′	$1/\sqrt{2}$	$1/\sqrt{2}$	0
Y′	$-1/\sqrt{2}$	$1/\sqrt{2}$	0
Z′	0	0	1

By the Equations (3.2a-f) and (3.3a-f), we have

$$\Im'_1 = \frac{1}{2}(\Im_1+2\Im_6+\Im_2) \quad \Im'_2=\frac{1}{2}(\Im_1-2\Im_6+\Im_2)$$

$$\Im'_3 = \Im_3 \quad \Im'_4=\frac{1}{\sqrt{2}}(\Im_4-\Im_5)$$

$$\Im'_5 = \frac{1}{\sqrt{2}}(\Im_4+\Im_5) \quad \Im'_6=\frac{1}{\sqrt{2}}(\Im_2-\Im_1) \qquad 6.7$$

and,

$$e'_1 = \frac{1}{2}(e_1+2e_6+e_2) \quad e'_2 = \frac{1}{2}(e_1-2e_6+e_2)$$

$$e'_3 = e_3 \quad e'_4 = \frac{1}{\sqrt{2}}(e_4-e_5)$$

$$e'_5 = \frac{1}{\sqrt{2}}(e_4+e_5) \quad e'_6 = (e_2-e_1) \qquad 6.8$$

From Equations (6.7) and (6.8), we have

$$\Im'_6 = \frac{1}{2}(\Im_2-\Im_1)$$

or $C_{66}e'_6 = \frac{1}{2}(C_{12}\, e_1 + C_{11}\, e_2 + C_{12}\, e_3 - C_{11}\, e_1$

$- C_{12}\, e_2 - C_{12}\, e_3)$

or $C_{66}(e_2-e_1) = \frac{1}{2}[(C_{12} - C_{11})e_1-(C_{12}-C_{11})e_2]$

Comparing the coefficients of e_1 (or e_2)

$C_{66}=\frac{1}{2}(C_{11}-C_{12})$.

Then, the system of Equation (6.6) reduces to

$$\begin{bmatrix}\Im_1\\ \Im_2\\ \Im_3\\ \Im_4\\ \Im_5\\ \Im_6\end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0\\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0\\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0\\ 0 & 0 & 0 & (C_{11}-C_{12})/2 & 0 & 0\\ 0 & 0 & 0 & 0 & (C_{11}-C_{12})/2 & 0\\ 0 & 0 & 0 & 0 & 0 & (C_{11}-C_{12})/2 \end{bmatrix} \begin{bmatrix} e_1\\ e_2\\ e_3\\ e_4\\ e_5\\ e_6\end{bmatrix} \qquad 6.9$$

There are only two elastic constants C_{11} and C_{12} in the stress-strain relations. Such a medium in which the elastic property is represented by only two elastic constants is called isotropic medium. Generally, in such a medium C_{11} = □ +2□ μ□ and C_{12} = □ . The stress-strain relation is represented by

$$\Im_{ij} = \lambda\Theta\delta_{ij} + 2\mu e_{ij}, \; (i, j = 1, 2, 3) \qquad 6.10$$

where Θ represents the cubical dilation, λ & μ are Lame's constants, δ_{if} represents the Kronecker's delta.

TRANSVERSE ISOTROPY AND AZIMUTHAL ANISOTROPY

The transverse isotropy is elastically equivalent to hexagonal symmetry with vertical axis symmetry. Keeping the Z-axis fixed and rotated the two axes at an angle Θ. The stress-strain relations remain invariant under this transformation. The direction cosines for the transformation are given by

	X	Y	Z
X'	Cos Θ	Sin Θ	0
Y'	-Sin Θ	Cos Θ	0
Z'	0	0	1

By the Equations (3.2a-f) and (3.3a-f), we have

$$\Im'_1 = \cos^2\Theta\, \Im_1 + 2\sin\Theta\cos\Theta\, \Im_6 + \sin^2\Theta\, \Im_2$$
$$\Im'_2 = \sin^2\Theta\, \Im_1 - 2\sin\Theta\cos\Theta\, \Im_6 + \cos^2\Theta\, \Im_2$$
$$\Im'_3 = \Im_3$$
$$\Im'_4 = -\sin\Theta\, \Im_5 + \cos\Theta\, \Im_4$$
$$\Im'_5 = \cos\Theta\, \Im_5 + \sin\Theta\, \Im_4$$
$$\Im'_6 = -\sin\Theta\cos\Theta\, \Im_1 + \cos^2\Theta\, \Im_6 - \sin^2\Theta\, \Im_6 + \sin\Theta\cos\Theta\, \Im_2 \qquad 7.1$$

and,

$$e'_1 = \cos^2\Theta e_1 + \sin\Theta\cos\Theta e_6 + \sin^2\Theta e_2$$
$$e'_2 = \sin^2\Theta e_1 - \sin\Theta\cos\Theta e_6 + \cos^2\Theta e_2$$
$$e'_3 = e_3$$
$$e'_4 = -\sin\Theta e_5 + \cos\Theta e_4$$
$$e'_5 = \cos\Theta e_5 + \sin\Theta e_4$$
$$e'_6 = -2\sin\Theta\cos\Theta e_1 + \cos^2\Theta e_6 - \sin^2\Theta e_6 + 2\sin\Theta\cos\Theta e_2 \qquad 7.2$$

Using Equations (5.3), (7.1) and (7.2), we get

$$\Im'_1 = \cos^2\Theta\, \Im_1 + 2\sin\Theta\cos\Theta\, \Im_6 + \sin^2\Theta\, \Im_2$$

or $C_{11}e'_1 + C_{12}e'_2 + C_{13}e'_3$

$= \cos^2\theta\ [C_{11}e_1 + C_{12}e_2 + C_{13}e_3]\ +2\sin\theta\cos\theta\ C_{66}e_6 + \sin^2[C_{21}e_1 + C_{22}e_2 + C_{23}e_3]$

or $C_{11}[\cos^2\theta\ e1 + \sin\theta\cos\theta\ e_6 + \sin^2\theta e_2] + C_{12}\ [\sin^2\theta\ e_1 - \sin\theta\cos\theta\ e_6 + \cos^2\theta e_2] + C_{13}e_3$

$= \cos^2[C_{11}e_1 + C_{12}e_2 + C_{13}e_3] + 2\sin\theta\cos\theta\ C_{66}e_6 + \sin^2\theta[C_{21}e_1 + C_{22}e_2 + C_{23}e_3]$

or $(C_{11}\cos^2\theta + C_{12}\sin^2\theta)e_1 + (C_{11}\sin^2\theta + C_{12}\cos^2\theta)e_2 + C_{13}e_3 + (C_{11}\text{-}C_{12})\sin\theta\cos\theta e_6$

$= (C_{11}\cos^2\theta + C_{12}\sin^2\theta)e_1 + (C_{12}\cos^2\theta + C_{22}\sin^2\theta)e_2 + (C_{13}\cos^2\theta + C_{23}\sin^2\theta)e_3 + 2\sin\theta\cos\theta C_{66}e_6$

Comparing the coefficients of e_2 (0r e_3) and e_6

$C_{11}\sin^2\theta + C_{12}\cos^2\theta = C_{12}\cos^2\theta + C_{22}\sin^2\theta$

which gives $C_{11} = C_{22}$

and $2\sin\theta\cos\theta C_{66} = \sin\theta\cos\theta(C_{11} - C_{12})$

or $C_{66} = (C_{11} - C_{12})/2$ which gives $C_{13} = C_{13}\cos^2\theta + C_{23}\sin^2\theta$ or $C_{13} = C_{23}$

By the relation $\Im'_4 = -\sin\theta\ \Im_5 + \cos\theta\Im_4$ we have

$C_{44}e'_4 = -\sin\theta\ C_{55}\ e_5 + \cos\theta\ C_{44}\ e_4$

or $C_{44}\ [-\sin\theta e_5 + \cos\theta e_4] = (C_{44}\cos\theta)\ e_4 - (C_{55}\sin\theta)e_5$

or $(C_{44}\cos\theta)e_4 - (C_{44}\sin\theta)e_5 = (C_{44}\cos\theta)\ e_4 - (C_{55}\sin\theta)e_5$

or $C_{44} = C_{55}$

Thus, under this symmetry

$C_{11} = C_{22}$; $C_{44} = C_{55}$; $C_{13} = C_{23}$; $C_{66} = (C_{11} - C_{12})/2$

Thus, in the crystallographic system, the stress-strain relations are given by

$$\begin{pmatrix} \Im_1 \\ \Im_2 \\ \Im_3 \\ \Im_4 \\ \Im_5 \\ \Im_6 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & (C_{11} - C_{12})/2 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix} \quad 7.3$$

There are only five elastic constants in the stress-strain relation. A medium in which the stress-strain relations are represented by only five elastic constants, is known as transverse isotropic (also known as radially anisotropic, axisymmetric and cylindrically symmetric) medium. A transversely isotropic material can be characterized by five independent elastic coefficients C_{11}, C_{33}, C_{13}, C_{44}, C_{66} that represent its aggregate properties.

CONCLUSION

We have discussed different elastic constants in the anisotropic elastic solids. We may conclude the following points:

1. The number of elastic constants in the generalized anisotropic medium is 21.
2. The number of elastic constants in the monoclinic medium is 13.
3. The number of elastic constants in the orthotropic medium is 9.
4. The number of elastic constants in the anisotropic medium with hexagonal symmetry is 6.
5. The number of elastic constants in the transversely isotropic medium is 5.
6. The number of elastic constants in the cubic medium is 3.
7. The number of elastic constants in the isotropic medium is 2.

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