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On a type of Para-Sasakian manifold

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ABSTRACT

In the present paper, some geometrical properties of projective curvature tensor, conformal curvature tensor and m-projective curvature tensor in Para-Sasakian manifold are studied.

Key words: Projective curvature tensors; conformal curvature tensor; contact manifolds.

Mathematical subject classification: 53C25

INTRODUCTION

Let M be a n-dimensional contact manifold with contact form η, i.e. $\eta \wedge (d\eta)^n \neq 0$. It is well known that a contact manifold admits a vector field ξ, called the characteristic vector field, such that $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for every $X \in \chi(M)$. Moreover, M admits a Riemannian metric g and a tensor field φ of type (1, 1) such that

$$\varphi^2 = I - \eta \otimes \xi, \quad g(X, \xi) = \eta(X), \quad g(\varphi X, Y) = d\eta(X, Y) \tag{1.a}$$

We then say that (φ, ξ, η, g) is a contact metric structure. A contact metric manifold is said to be a Sasakian if

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X \tag{1.b}$$

In which case

$$(\nabla_X \xi) = -\varphi X, \quad R(X, Y)\xi = \eta(Y)X - \eta(X)Y \tag{1.c}$$

where ∇ denotes the operator of covariant ifferentiation with respect to the metric tensor g.

An n-dimensional differential manifold M is said to admit an almost paracontact Riemannian structure (φ, ξ, η, g) on M if

$$\varphi\xi = 0, \quad \eta\varphi = 0, \quad \eta(\xi) = 1, \quad g(X, \xi) = \eta(X), \tag{1.d}$$

$$\varphi^2 X = X - \eta(X)\xi, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.e}$$

for all vector fields X and Y on M. If (φ, ξ, η, g) satisfy the equation

$$\nabla_X \xi = \varphi X, \tag{1.f}$$

$$(\nabla_X \varphi)Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{1.g}$$

then M is called a Para-Sasakian manifold.

It is well known that in a Para-Sasakian manifold the following relations hold [2]:

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{1.1}$$

$$S(X, \xi) = -(n-1)\eta(X) \tag{1.2}$$

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$R(X,YZ) = g(X,Z)Y - g(Y,Z)X$ (1.3)
$R(X,Y,\xi) = \eta(X)Y - \eta(Y)X$ (1.4)
$S(\varphi X,\varphi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y)$ (1.5)
where S is Ricci tensor and R is the curvature tensor of the manifold M.

THEOREM

(1): In a Para-Sasakian manifold with constant curvature -1, we have

$$P'(\varphi X,\varphi Y,\varphi Z,\varphi W) = P'(X,Y,Z,W) - \eta(W)P'(X,Y,Z,\xi)$$

where P is the projective curvature and given as

$$P(X,Y,Z) = R(X,Y,Z) - \frac{1}{n-1}\{S(Y,Z)X - S(X,Z)Y\}. \quad (1.6)$$

PROOF

We know that
$P'(X,Y,Z,W) = g(P(X,Y,Z)W)$
and therefore from (1.6), we have

$$P'(X,Y,Z,W) = R'(X,Y,Z,W) - \frac{1}{n-1}\{S(Y,Z)g(X,W) - S(X,Z)g(Y,W)\} \quad (1.7)$$

Operating φ on X, Y, Z and W on both sides of (1.7), we have

$$P'(\varphi X,\varphi Y,\varphi Z,\varphi W) = R'(\varphi X,\varphi Y,\varphi Z,\varphi W) - \frac{1}{n-1}\{S(\varphi Y,\varphi Z)g(\varphi X,\varphi W) - S(\varphi X,\varphi Z)g(\varphi Y,\varphi W)\}$$

Using equations (1.1), (1.3) and (1.5) in above equation, we have

$$P'(\varphi X,\varphi Y,\varphi Z,\varphi W) = [\{g(X,Z)-\eta(X)\eta(Z)\}\{g(Y,W)-\eta(Y)\eta(W)\} - \{g(X,Z)-\eta(X)\eta(Z)\}\{g(X,Z)-\eta(X)\eta(Z)\}] - \frac{1}{n-1}[\{S(Y,Z)+(n-1)\eta(Y)\eta(Z)\}\{g(X,W)-\eta(X)\eta(W)\} - \{S(X,Z)+(n-1)\eta(X)\eta(Z)\}\{g(Y,W)-\eta(Y)\eta(W)\}]$$

$$= \{g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\} - \frac{1}{n-1}\{S(Y,Z)g(X,W) - S(X,Z)g(Y,W)\} - \eta(W)[g(X,Z)\eta(Y) - g(Y,Z)\eta(X) - \frac{1}{n-1}\{S(Y,Z)\eta(X) - S(X,Z)\eta(Y)\}]$$

$$R'(X,Y,Z,W) - \frac{1}{n-1}\{S(Y,Z)g(X,W) - S(X,Z)g(Y,W)\} - \eta(W)[g(X,Z)\eta(Y) - g(Y,Z)\eta(X) - \frac{1}{n-1}\{S(Y,Z)\eta(X) - S(X,Z)\eta(Y)\}].$$

Therefore, we have

$$P'(\varphi X,\varphi Y,\varphi Z,\varphi W) = P'(X,Y,Z,W) - \eta(W)P'(X,Y,Z,\xi).$$

This completes the proof.

THEOREM

(2): In a Para-Sasakian manifold with constant curvature -1, we have

$$C'(\varphi X,\varphi Y,\varphi Z,\varphi W) = C'(X,Y,Z,W) - \eta(W)C'(X,Y,Z,\xi) + \eta(Z)C'(X,Y,W,\xi)$$

where C is the Conformal curvature tensor and given as

$$C(X,Y,Z) = R(X,Y,Z) - \frac{1}{n-2}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} + \frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\}. \quad (1.8)$$

PROOF

From (1.8), we have

$$C'(X,Y,Z,W) = R'(X,Y,Z,W) - \frac{1}{n-2}\{S(Y,Z)g(X,W) - S(X,Z)g(Y,W) + g(Y,Z)S(X,W) - g(X,Z)S(Y,W)\} + \frac{r}{(n-1)(n-2)}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}$$
(1.9)

Operating φ on X, Y, Z and W on both sides of (1.9), we have

$$C'(\varphi X,\varphi Y,\varphi Z,\varphi W) = R'(\varphi X,\varphi Y,\varphi Z,\varphi W) - \frac{1}{n-2}\{S(\varphi Y,\varphi Z)g(\varphi X,\varphi W) - S(\varphi X,\varphi Z)g(\varphi Y,\varphi W) + g(\varphi Y,\varphi Z)S(\varphi X,\varphi W) - g(\varphi X,\varphi Z)S(\varphi Y,\varphi W)\}$$

$$+\frac{r}{(n-1)(n-2)}\{g(\varphi Y,\varphi Z)g(\varphi X,\varphi W) - g(\varphi X,\varphi Z)g(\varphi Y,\varphi W)\}$$

Using equations (1.1), (1.3) and (1.5) in above equation, we have

$$C'(\varphi X,\varphi Y,\varphi Z,\varphi W) = [\{g(X,Z)-\eta(X)\eta(Z)\}\{g(Y,W)-\eta(Y)\eta(W)\}$$
$$-\{g(Y,Z)-\eta(Y)\eta(Z)\}\{g(X,W)-\eta(X)\eta(W)\}]$$
$$-\frac{1}{n-2}[\{S(Y,Z)+(n-1)\eta(Y)\eta(Z)\}\{g(X,W)-\eta(X)\eta(W)\}$$
$$-\{S(X,Z)+(n-1)\eta(X)\eta(Z)\}\{g(Y,W)-\eta(Y)\eta(W)\}$$
$$+\{g(X,Z)-\eta(X)\eta(Z)\}\{S(Y,W)+(n-1)\eta(Y)\eta(W)\}$$
$$-\{g(Y,Z)-\eta(Y)\eta(Z)\}\{S(X,W)+(n-1)\eta(X)\eta(W)\}]$$
$$+\frac{r}{(n-1)(n-2)}[\{g(Y,Z)-\eta(Y)\eta(Z)\}\{g(X,W)-\eta(X)\eta(W)\}$$
$$-\{g(X,Z)-\eta(X)\eta(Z)\}\{g(Y,W)-\eta(Y)\eta(W)\}]$$
$$=\{g(X,Z)g(Y,W)-g(X,W)g(Y,Z)\}-\eta(W)\{g(X,Z)\eta(Y)-g(Y,Z)\eta(X)\}+\eta(Z)\{g(X,W)\eta(Y)-g(Y,W)\eta(X)\}$$
$$-\frac{1}{n-2}[S(Y,Z)g(X,W)-S(X,Z)g(Y,W)+g(Y,Z)S(X,W)-g(X,Z)S(Y,W)-S(X,Z)\eta(Y)\eta(W)+S(Y,Z)\eta(X)\eta(W)$$
$$-(n-1)g(Y,Z)\eta(X)\eta(W)+(n-1)g(X,Z)\eta(Y)\eta(W)-S(Y,Z)\eta(X)\eta(W)+S(X,W)\eta(Y)\eta(Z)-(n-1)g(Y,W)\eta(X)\eta(Z)$$
$$+(n-1)g(Y,W)\eta(X)\eta(Z)]$$
$$+\frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)+g(X,Z)\eta(Y)\eta(W)-g(Y,Z)\eta(X)\eta(W)$$
$$+g(Y,W)\eta(X)\eta(Z)-g(X,W)\eta(Y)\eta(Z)].$$

After some calculations, we have

$$C'(\varphi X,\varphi Y,\varphi Z,\varphi W) = C'(X,Y,Z,W) - \eta(W)C'(X,Y,Z,\xi) + \eta(Z)C'(X,Y,W,\xi)$$

This completes proof of the theorem.

Theorem

(3): In a Para-Sasakian manifold with constant curvature -1, we have

$$H'(\varphi X,\varphi Y,\varphi Z,\varphi W) = H'(X,Y,Z,W) - \eta(W)H'(X,Y,Z,\xi) + \eta(Z)H'(X,Y,W,\xi)$$

Where m - projective curvature tensor is given by[3]

$$H(X,Y,Z) = R(X,Y,Z) - \frac{1}{2(n-1)}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} \quad (1.10)$$

and $H'(X,Y,Z,W) = H'(Z,W,X,Y)$

Proof

From (1.10), we have

$$H'(X,Y,Z,W) = R'(X,Y,Z,W) - \frac{1}{2(n-1)}\{S(Y,Z)g(X,W) - S(X,Z)g(Y,W) + g(Y,Z)S(X,W) - g(X,Z)S(Y,W)\} \quad (1.11)$$

Operating φ on X, Y, Z and W on both sides of (1.11), we have

$$H'(\varphi X,\varphi Y,\varphi Z,\varphi W) = R'(\varphi X,\varphi Y,\varphi Z,\varphi W) - \frac{1}{2(n-1)}\{S(\varphi Y,\varphi Z)g(\varphi X,\varphi W) - S(\varphi X,\varphi Z)g(\varphi Y,\varphi W) + g(\varphi Y,\varphi Z)S(\varphi X,\varphi W) - g(\varphi X,\varphi Z)S(\varphi Y,\varphi W)\}.$$

Using equations (1.1), (1.3) and (1.5) in above equation, we have

$$H'(\varphi X,\varphi Y,\varphi Z,\varphi W) = [\{g(X,Z)-\eta(X)\eta(Z)\}\{g(Y,W)-\eta(Y)\eta(W)\}$$
$$-\{g(Y,Z)-\eta(Y)\eta(Z)\}\{g(X,W)-\eta(X)\eta(W)\}]$$

$$\frac{1}{n-2}[\{S(Y,Z)+(n-1)\eta(Y)\eta(Z)\}\{g(X,W)-\eta(X)\eta(W)\}$$
$$-\{S(X,Z)+(n-1)\eta(X)\eta(Z)\}\{g(Y,W)-\eta(Y)\eta(W)\}$$
$$+\{g(X,Z)-\eta(X)\eta(Z)\}\{S(Y,W)+(n-1)\eta(Y)\eta(W)\}$$
$$-\{g(Y,Z)-\eta(Y)\eta(Z)\}\{S(X,W)+(n-1)\eta(X)\eta(W)\}]$$
$$=\{g(X,Z)g(Y,W)-g(X,W)g(Y,Z)\}-\eta(W)\{g(X,Z)\eta(Y)-g(Y,Z)\eta(X)\}+\eta(Z)\{g(X,W)\eta(Y)-g(Y,W)\eta(X)\}$$
$$-\frac{1}{n-2}[S(Y,Z)g(X,W)-S(X,Z)g(Y,W)+g(Y,Z)S(X,W)-g(X,Z)S(Y,W)-S(X,Z)\eta(Y)\eta(W)+S(Y,Z)\eta(X)\eta(W)$$
$$-(n-1)g(Y,Z)\eta(X)\eta(W)+(n-1)g(X,Z)\eta(Y)\eta(W)-S(Y,Z)\eta(X)\eta(W)+S(X,W)\eta(Y)\eta(Z)-(n-1)g(Y,W)\eta(X)\eta(Z)$$
$$+(n-1)g(Y,W)\eta(X)\eta(Z)]$$
$$=R'(X,Y,Z,W)-\frac{1}{2(n-1)}\{S(Y,Z)g(X,W)-S(X,Z)g(Y,W)+g(Y,Z)S(X,W)-g(X,Z)S(Y,W)\}$$
$$-\eta(W)[g(X,Z)\eta(Y)-g(Y,Z)\eta(X)-\frac{1}{2(n-1)}\{S(Y,Z)\eta(X)-S(X,Z)\eta(Y)+(n-1)g(X,Z)\eta(Y)$$
$$-(n-1)g(Y,Z)\eta(X)\}]+\eta(Z)[g(X,W)\eta(Y)-g(Y,W)\eta(X)$$
$$-\frac{1}{2(n-1)}\{S(Y,W)\eta(X)-S(X,W)\eta(Y)+(n-1)g(X,W)\eta(Y)$$
$$+(n-1)g(X,W)\eta(Y)\ -(n-1)g(Y,W)\eta(X)\}].$$

Therefore, finally we have

$$H'(\varphi X,\varphi Y,\varphi Z,\varphi W)=H'(X,Y,Z,W)-\eta(W)H'(X,Y,Z,\xi)+\eta(Z)H'(X,Y,W,\xi)$$

This completes proof of the theorem.

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