MIZO POST GRADUATE SCIENCE SOCIETY MIPOGRASS
www.sciencevision.org
Research Note
Sci Vis Vol 12 Issue No 4
October-December 2012
ISSN (print) 0975-6175
ISSN (online) 2229-6026

On a generalized almost contact metric normal manifold

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Received 25 September 2012 | Revised 25 November 2012 | Accepted 29 November 2012

Abstract

The present paper deals with the study of the geometrical properties of some almost contact metric manifolds such as generalized almost contact metric normal manifold and nearly M-manifolds. Almost contact manifolds play an important role in differential geometry. The author has proved that a generalized almost contact metric normal manifold admits nearly M-manifold.

Key words: Almost contact metric manifolds; nearly M- manifold; affine connection.

Introduction

Consider an odd dimensional differential manifold of differentiability class C^∞ - on which there are defined a tensor field F of type (1,1), a vector field T and a 1-form A satisfying for arbitrary vector fields X,Y,Z

(a) $\overline{\overline{X}} + X = A(X)T$

(b) $\overline{X} \underline{\underline{\text{def}}} F(X)$

(c) $A(\overline{X}) = 0$

(d) $A(T) = 1$, (1.1)

then M2m$_{+1}$ is said to be an almost contact manifold.

An almost contact manifold M_{2m+1} on which a metric tensor g satisfying

(a) $'F(X,Y) = g(\overline{X},Y) = -g(X,\overline{Y})$

(b) $g(X,T) = A(X)$, (1.2)

is called an almost contact metric manifold with structure {F, T, A, g} [1].

If on an almost contact metric manifold T satisfying

(a) $(D_X A)(\overline{Y}) = (D_{\overline{X}} A)(Y) = -(D_Y A)(\overline{X}) \Leftrightarrow$

$(D_X A)(Y) = -(D_Y A)(X) = -(D_{\overline{X}} A)(\overline{Y})$

and (b) $D_T F = 0$ (1.3)

then T is said to be of second class and the manifold is said to be of second class.

The almost contact metric manifolds satisfying [3]

$$(D_{\overline{X}}\,'F)(\overline{Y},Z) - (D_X\,'F)(Y,Z) = 2A(Z)\ (D_X A)(\overline{Y}) - A(Y)(D_X A)(\overline{Z}) \quad (1.4)$$

$$(D_X\,'F)(Y,Z) + (D_Y\,'F)(X,Z) = A(Y)\ (D_{\overline{X}} A)(Z) + A(X)(D_{\overline{Y}} A)(Z) \quad (1.5)$$

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where D is Riemannian connection, are called generalized almost contact metric normal manifold and nearly M-manifold respectively.

An almost contact metric manifold is said to be quasi-Sasakian manifold, if

$$d'F = 0 \Leftrightarrow (D_X{}'F)(Y,Z) + (D_Y{}'F)(Z,X) + (D_Z{}'F)(X,Y) = 0 \tag{1.6}$$

On an almost contact metric manifold following hold [2].

(a) $(D_X A)(Y) = g(D_X T, Y)$

(b) $(D_X{}'F)(Y,T) = -(D_X A)(\bar{Y})$

(c) $(D_X{}'F)(\bar{Y},\bar{Z}) + (D_X{}'F)(Y,Z) = A(Y)(D_X A)(\bar{Z}) - A(Z)\,(D_X A)\,(\bar{Y})$

(d) $(D_X{}'F)(\bar{Y},Z) - (D_X{}'F)(Y,\bar{Z}) = A(Z)\,(D_X A)(Y) + A(Y)(D_X A)(Z)$ (1.7)

Nijenhuis tensor with respect to F is a vector valued, bilinear function N, given by

$$N(X,Y) = (D_{\bar{X}}F)(Y) - (D_{\bar{Y}}F)(X) - \overline{(D_X F)(Y)} + \overline{(D_Y F)(X)} \tag{1.8}$$

Let us assume that

(a) $M_1(X,Y) = D_{\bar{X}}\bar{Y} - D_X Y - \overline{D_{\bar{X}} Y} - \overline{D_X \bar{Y}} + A(D_X Y)T$

Or

(b) $M_1(X,Y) = (D_{\bar{X}}F)(Y) - \overline{(D_X F)(Y)}$ (1.9)

An almost contact metric manifold is said to be integrable, if

$${}'N(\bar{X},\bar{Y},\bar{Z}) = 0$$

Generalized almost contact metric normal manifold

Theorem (2.1): On a generalized almost contact metric normal manifold, we have

(a) ${}'F(Y, D_{\bar{X}}T - \overline{D_X T}) + 2(D_X A)(Y) = 0$

(b) $N(\bar{X},\bar{\bar{Y}}) - N(\bar{Y},\bar{\bar{X}}) = 4(D_X A)(\bar{Y})T$ (2.1)

Proof: Putting T for Z in (1.5), we get

$$(D_{\bar{X}}{}'F)(\bar{Y},T) - (D_X{}'F)(Y,T) = 2(D_{\bar{X}}A)(Y)$$

Or

$$\bar{X}({}'F(\bar{Y},T)) - {}'F(D_{\bar{X}}\bar{Y},T) - {}'F(\bar{Y},D_{\bar{X}}T) - X({}'F(Y,T)) + {}'F(D_X Y,T) + {}'F(Y,D_X T) = 2(D_{\bar{X}}A)(Y)$$

By using ${}'F(X,T) = 0$ in above, we get

$$-{}'F(\bar{Y},D_{\bar{X}}T) + {}'F(Y,D_X T) = 2(D_{\bar{X}}A)(Y)$$

Or

$$-g(\bar{\bar{Y}},D_{\bar{X}}T) - g(Y,\overline{D_X T}) = 2(D_{\bar{X}}A)(Y)$$

$$\Rightarrow g(Y, D_{\bar{X}}T - \overline{D_X T}) = 2(D_{\bar{X}}A)(Y) + A(Y)\,A(D_{\bar{X}}T)$$

Barring Y in above equation, we get (2.1) (a)

Now, from (1.4) we have

$$(D_{\bar{X}}{}'F)(\bar{Y}) - (D_X F)(Y) = 2(D_{\bar{X}}A)(Y)T + A(Y)\overline{(D_X T)} \tag{2.2}$$

Barring Y in (2. 2), we obtain

$$(D_{\bar{X}}F)(Y) + (D_X F)(\bar{Y}) + A(Y)\overline{(D_{\bar{X}}T)} = 2(D_X A)(Y)T \tag{2.3}$$

Now, from (1.8) we have

$$N(X,\bar{Y}) = (D_X F)(\bar{Y}) + (D_Y F)(X) - \overline{(D_X F)(\bar{Y})} + \overline{(D_{\bar{Y}}F)(X)} \tag{2.4}$$

thus, from above equation, we have

$$N(X,\bar{Y}) - N(Y,\bar{X}) = (D_{\bar{X}}F)(\bar{Y}) - (D_X F)(Y) - (D_{\bar{Y}}F)(\bar{X}) + (D_Y F)(X)$$

$$-\overline{(D_X F)(\bar{Y})}+\overline{(D_{\bar{Y}} F)(X)}$$
$$+\overline{(D_Y F)(\bar{X})}-\overline{(D_{\bar{X}} F)(Y)} \tag{2.5}$$

Using (1.3), (2. 2), (2. 3) in equation (2. 5), we obtain

$$N(X,\bar{Y})-N(Y,\bar{X})=2(D_{\bar{X}}A)(Y)T$$
$$+A(Y)\left(\overline{D_X T}\right)-2(D_{\bar{Y}}A)(X)T$$
$$-A(X)\left(\overline{D_Y T}\right)+A(Y)\left(\overline{\overline{D_{\bar{X}} T}}\right)-A(X)\left(\overline{\overline{D_{\bar{Y}} T}}\right)$$

Or

$$N(X,\bar{Y})-N(Y,\bar{X})=A(Y)\left(D_X T+\overline{\overline{D_{\bar{X}} T}}\right)$$
$$-A(X)\left(\overline{D_Y T}+\overline{\overline{D_{\bar{Y}} T}}\right)$$
$$+4(D_X A)(\bar{Y})T \tag{2. 6}$$

Barring X and Y in equation (2. 6) and using (1.3), we get ((2.1) b)

Theorem 2.2: A generalized almost contact metric normal manifold admits nearly M-manifold.

Proof: From (1.4), we have

$$(D_{\bar{X}}\,{}'F)\,(\bar{Y},Z)-(D_X\,{}'F)(Y,Z)=2$$
$$A(Z)(D_X A)(\bar{Y})-A(Y)(D_X A)(\bar{Z}) \tag{2.7}$$

Interchanging X and Y, we get

$$(D_{\bar{Y}}\,{}'F)(\bar{X},Z)-(D_Y\,{}'F)(X,Z)=$$
$$2A(Z)(D_Y A)(\bar{X})-A(X)(D_Y A)(\bar{Z}) \tag{2.8}$$

Adding (2.7) and (2.8) and using (1.5), we get

$$(D_X\,{}'F)(Y,Z)+(D_Y\,{}'F)(X,Z)=$$
$$A(Y)(D_{\bar{X}}A)(Z)+A(X)(D_{\bar{Y}}A)(Z)$$

which complete the proof.

Affine Connection

Let B be an affine connection in an almost contact metric manifold. Put

$$B_X Y=D_X Y+H(X,Y)\,. \tag{3.1}$$

The Torsion tensor S of B is given by

$$S(X,Y)=H(X,Y)-H(Y,X) \tag{3.2}$$

where

$${}'S(X,Y,Z)={}'H(X,Y,Z)-{}'H(Y,X,Z) \tag{3.3}$$

where

$${}'S(X,Y,Z)=g(S(X,Y),Z) \quad \text{and}$$
$${}'H(X,Y,Z)=g(H(X,Y),Z).$$

Theorem 3.1: On a generalized almost contact metric normal manifold, we have

$$2A(Z)\left[(B_T A)(\bar{Y})+{}'H(T,\bar{Y},T)\right]$$
$$=A(Y)\left[(B_T A)(\bar{Z})+{}'H(T,\bar{Z},T)\right]. \tag{3.4}$$

Proof: From ((1.3) b), we have

$$D_T F=0, \text{ which implies}$$
$$T\,{}'F(X,Y)={}'F(D_T X,Y)+{}'F(X,D_T Y)$$
$$=(B_T\,{}'F)(X,Y)+{}'F(B_T X,Y)+{}'F(X,B_T Y)$$

Or

$$(B_T\,{}'F)(X,Y)={}'H(T,X,\bar{Y})-{}'H(T,Y,\bar{X}) \tag{3.5}$$

Similarly, we have

$$(D_X\,{}'F)(Y,Z)=(B_X\,{}'F)(Y,Z)$$
$$-{}'H(X,Y,\bar{Z})+{}'H(X,Z,\bar{Y}) \tag{3.6}$$

Barring X and Y in above equation, we get

$$(D_{\bar{X}}\,{}'F)(\bar{Y},Z)=(B_{\bar{X}}\,{}'F)(\bar{Y},Z)$$
$$-{}'H(\bar{X},\bar{Y},\bar{Z})+{}'H(\bar{X},Z,\bar{\bar{Y}}) \tag{3.7}$$

Also, we have

$$A(Y)(D_X A)(\bar{Z})=$$
$$A(Y)\left[(D_X A)(\bar{Z})+{}'H(X,\bar{Z},T)\right] \tag{3.8}$$

and

$$A(Z)\ (D_X A)(\bar{Y}) =$$
$$A(Z)\ \left[(B_X A)(\bar{Y}) + {}'H(X,\bar{Y},T)\right] \quad (3.9)$$

Putting these values in equation (1.4), we have

$$(B_{\bar{X}}{}'F)(\bar{Y},Z) - (B_X{}'F)(Y,Z) - {}'H(\bar{X},\bar{Y},\bar{Z})$$
$$+{}'H(\bar{X},Z,\bar{\bar{Y}}) + {}'H(X,Y,\bar{Z}) - {}'H(X,Z,\bar{Y})$$
$$= 2A(Z)[(B_X A)(\bar{Y}) + {}'H(X,\bar{Y},T)]$$
$$- A(Y)[(B_X A)(\bar{Z}) + {}'H(X,\bar{Z},T)] \quad (3.10)$$

Putting X = T and using (3.4) in equation (3.9), we get the equation (3.1.1).

Theorem 3.2 When B satisfies

(a) $B_X{}'F = 0$

(b) ${}'H(\bar{X},\bar{Y},\bar{\bar{Z}}) + {}'H(\bar{X},\bar{Z},\bar{\bar{Y}}) = 0$ (3.11)

and generalized almost contact metric normal manifold is integrable, then

$$4{}'H(\bar{X},\bar{Y},\bar{Z}) = 2(D_{\bar{X}}{}'F)(\bar{Z},\bar{Y})$$
$$+(D_{\bar{Z}}{}'F)(\bar{\bar{Y}},\bar{X}) - (D_{\bar{Y}}{}'F)(\bar{\bar{Z}},\bar{X})$$
$$+(D_{\bar{Z}}{}'F)(\bar{\bar{X}},\bar{\bar{Y}}) - (D_{\bar{\bar{Y}}}{}'F)(\bar{\bar{X}},\bar{\bar{Z}}) \quad (3.12)$$

Proof: From (1.4), we have

$$(D_{\bar{X}}{}'F)(\bar{Y},Z) - (D_X{}'F)(Y,Z) =$$
$$2A(Z)(D_X A)(\bar{Y}) - A(Y)(D_X A)(\bar{Z}) \quad (3.13)$$

Similarly, writing two other equations by cyclic order of X, Y, Z we get

$$(D_{\bar{Y}}{}'F)(\bar{Z},X) - (D_Y{}'F)(Z,X) =$$
$$2A(X)(D_Y A)(\bar{Z}) - A(Z)(D_Y A)(\bar{X}) \quad (3.14)$$

and

$$(D_{\bar{Z}}{}'F)(\bar{X},Y) - (D_Z{}'F)(X,Y) =$$
$$2A(Y)(D_Z A)(\bar{X}) - A(X)(D_Z A)(\bar{Y}) \quad (3.15)$$

Adding (3.13), (3.14) and (3.15), we get

$$(D_{\bar{X}}{}'F)(\bar{Y},Z) + (D_{\bar{Y}}{}'F)(\bar{Z},X)$$
$$+(D_{\bar{Z}}{}'F)(\bar{X},Y) - (d{}'F)(X,Y,Z)$$
$$= 2\begin{bmatrix} A(X)(D_{\bar{Y}}A)(Z) + A(Y)(D_{\bar{Z}}A)(X) \\ +A(Z)(D_{\bar{X}}A)(Y) \end{bmatrix}$$
$$-\begin{bmatrix} A(X)(D_Z A)(\bar{Y}) + A(Y)(D_X A)(\bar{Z}) + \\ A(Z)(D_X A)(\bar{X}) \end{bmatrix}$$

Using (1.3) in the above equation, we obtain

$$(D_{\bar{X}}{}'F)(\bar{Y},Z) + (D_{\bar{Y}}{}'F)(\bar{Z},X)$$
$$+(D_{\bar{Z}}{}'F)(\bar{X},Y) - (d{}'F)(X,Y,Z)$$
$$= 3\begin{bmatrix} A(X)\ (D_{\bar{Y}}A)(Z) + A(Y)(D_{\bar{Z}}A)(X) \\ +A(Z)(D_{\bar{X}}A)(Y) \end{bmatrix} \quad (3.16)$$

thus, we have

$$(d{}'F)(\bar{X},\bar{\bar{Y}},\bar{Z}) = -(D_{\bar{X}}{}'F)(\bar{Y},\bar{Z})$$
$$-(D_{\bar{Y}}{}'F)(\bar{\bar{Z}},\bar{X}) + (D_{\bar{\bar{Z}}}{}'F)(\bar{\bar{X}},\bar{Y}) \quad (3.17)$$

Interchanging, Y and Z in above equation, we get

$$(d{}'F)\ (\bar{X},\bar{\bar{Z}},\bar{Y}) = -(D_{\bar{X}}{}'F)(\bar{Z},\bar{Y})$$
$$-(D_{\bar{z}}{}'F)(\bar{\bar{Y}},\bar{X}) + (D_{\bar{Y}}{}'F)(\bar{\bar{X}},\bar{\bar{Z}}) \quad (3.18)$$

We know that necessary condition for an almost contact manifold to be integrable is [2]

$$(d{}'F)(\bar{X},\bar{\bar{Y}},\bar{Z}) - (d{}'F)(\bar{X},\bar{\bar{Z}},\bar{Y})$$
$$= 4{}'H(\bar{X},\bar{Y},\bar{Z}) \quad (3.19)$$

Now, using (3.17) and (3.18) in equation (3.19), we obtain (3.12).

Theorem 3.3 When B satisfies

(a) $B_X{}'F = 0$

(b) $'H(X,Y,\bar{Z}) + 'H(Y,X,\bar{Z}) = 0$ (3.20)

and generalized almost contact metric normal manifold is completely integrable, then

$$2'H(\bar{X},\bar{Y},\bar{Z}) = (D_{\bar{X}}'F)(\bar{Y},\bar{\bar{Z}}) + (D_{\bar{Y}}'F)(\bar{Z},\bar{\bar{X}}) + (D_{\bar{Z}}'F)(\bar{X},\bar{\bar{Y}}). \quad (3.21)$$

Proof: We have [2]

$$2'H(\bar{X},\bar{Y},\bar{Z}) = (d'F)(\bar{\bar{X}},\bar{\bar{Y}},\bar{\bar{Z}}) \quad (3.22)$$

From equation (3.2.6), we have

$$(d'F)(\bar{X},\bar{Y},\bar{Z}) = (D_{\bar{\bar{X}}}'F)(\bar{\bar{Y}},\bar{Z}) + (D_{\bar{\bar{Y}}}'F)(\bar{\bar{Z}},\bar{X}) + (D_{\bar{\bar{Z}}}'F)(\bar{\bar{X}},\bar{Y}). \quad (3.23)$$

From equation (3.22) and (3.23) we get equation (3.21).

Theorem 3.4: A generalized almost contact metric normal manifold is a quasi-Sasakian manifold, if

$$(D_{\bar{X}}'F)(\bar{Y},Z) + (D_{\bar{Y}}'F)(\bar{Z},X) + (D_{\bar{Z}}'F)(\bar{X},Y) = 3\begin{bmatrix} 'M_1(X,\bar{Y},Z) + 'M_1(Y,\bar{Z},X) \\ + 'M_1(Z,\bar{X},Y) \end{bmatrix} \quad (3.24)$$

and

(a) $B_X'F = 0$ (b) $'H(X,Y,\bar{Z}) + 'H(Z,Y,\bar{X}) = 0$ (3.25)

Proof: From equation (1.9.b), we have

$$'M_1(X,Y,Z) = (D_{\bar{X}}'F)(Y,Z) + (D_X'F)(Y,\bar{Z})$$

Barring Y and using (1.4) in above, we obtain

$$'M_1(X,\bar{Y},Z) = A(Z)(D_X A)(\bar{Y})$$

In consequence of above equation, we have

$$'M_1(X,\bar{Y},Z) + 'M_1(Y,\bar{Z},X) + 'M_1(Z,\bar{X},Y)$$
$$= A(X)\ (D_Y A)(\bar{Z}) + A(Y)\ (D_Z A)(\bar{X}) + A(Z)\ (D_X A)(\bar{Y}) \quad (3.26)$$

Now using (3.25) in above equation, we get

$$(d'F)(X,Y,Z) = 2\begin{bmatrix} 'H(Z,Y\bar{X}) + 'H(X,Y,\bar{Z}) \\ + 'H(Y,X,\bar{Z}) \end{bmatrix} \quad (3.27)$$

From (3.16), we have

$$(D_{\bar{X}}'F)(\bar{Y},Z) + (D_{\bar{Y}}'F)(\bar{Z},X) + (D_{\bar{Z}}'F)(\bar{X},Y) - (d'F)(X,Y,Z)$$
$$= 3\begin{bmatrix} A(X)(D_{\bar{Y}}A)(Z) + A(Y)(D_{\bar{Z}}A)(X) \\ + A(Z)(D_{\bar{X}}A(Y)) \end{bmatrix} \quad (3.28)$$

From equations (3.27) and (3.28), we have

$$(D_{\bar{X}}'F)(\bar{Y},Z) + (D_{\bar{Y}}'F)(\bar{Z},X) + (D_{\bar{Z}}'F)(\bar{X},Y) - (d'F)(Z,X,Y)$$
$$= 3\begin{bmatrix} 'M_1(X,\bar{Y},Z) + 'M_1(X,\bar{Z},X) \\ + 'M_1(Z,\bar{X},Y) \end{bmatrix}. \quad (3.29)$$

The condition for an almost contact metric manifold to be quasi-Sasakian manifold is

$$(d'F)(X,Y,Z) = 0\ .$$

Thus in consequence of the above equation and (3.29), we get (3.24).

Theorem 3.5 On a generalized almost contact metric normal manifold, F is killing if

$$(D_{\bar{X}}'F)(\bar{Y},Z) + (D_{\bar{Y}}'F)(\bar{X},Z) = A(X)(D_Z A)(\bar{Y}) + A(Y)(D_Z A)(\bar{X}) \quad (3.30)$$

Proof: From (1.4), we have

$$(D_{\bar{X}}'F)(\bar{Y},Z) - (D_X'F)(Y,Z) = 2A(Z)(D_X A)(\bar{Y}) - A(Y)\ (D_X A)(\bar{Z}) \quad .(3.31)$$

Interchanging X and Y in above equation, we get (3.30)

Theorem 3.6: Let B satisfy

(a) $\left(B_X{}'F\right)\left(\overline{Y},\overline{Z}\right)=0$

(b) $'H\left(X,\overline{Y},\overline{\overline{Z}}\right)={}'H\left(X,\overline{Z},\overline{\overline{Y}}\right)$ (3.33)

then in a generalized almost contact metric normal manifold, we have

$$'H\left(\overline{X},\overline{\overline{Y}},\overline{\overline{Z}}\right)+{}'\overline{H}(\overline{X},\overline{Z},\overline{Y})=0 \tag{3.34}$$

Proof: Barring Y and Z in (3.10), we have

$$\left(B_{\overline{X}}{}'F\right)\left(\overline{\overline{Y}},\overline{Z}\right)-\left(B_X{}'F\right)\left(\overline{Y},\overline{Z}\right)$$
$$={}'H\left(\overline{X},\overline{\overline{Y}},\overline{\overline{Z}}\right)+{}'H\left(\overline{X},\overline{Z},\overline{Y}\right)$$
$$+{}'H\left(X,\overline{Z},\overline{\overline{Y}}\right)-{}'H\left(X,\overline{Y},\overline{\overline{Z}}\right) \tag{3.35}$$

Using (3.33) in equation (3.35), we get

$$'H\left(\overline{X},\overline{\overline{Y}},\overline{\overline{Z}}\right)+{}'H\left(\overline{X},\overline{Z},\overline{Y}\right)=0$$

this completes the proof.

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