http://sciencevision.info

Research Note

Sci Vis 10 (2), 62-65
April-June, 2010
ISSN 0975-6175

On a semi-symmetric metric connexion on a Riemannian manifold

J. P. Singh

Department of Mathematics and Computer Sciences, Mizoram University, Aizawl 796009, India

Received 8 July 2010 | Revised 14 July 2010 | Accepted 15 July 2010

ABSTRACT

Yano studied semi-symmetric connexion on a Riemannian manifold (M^n, g) and Thompson introduced 2-recurrent Riemannian manifold. The author here studied the nature of the 2-recurrent Riemannian manifold that admits a semi-symmetric connexion and it is found that such a manifold (M^n, g) is M-Projectively 2-recurrent.

Key words: 2-recurrent Riemannian manifold; M-Projectively curvature tensor; M-Projectively 2-recurrent.

Mathematical subject classification: 53C15

INTRODUCTION

This paper deals with a type of semi-symmetric metric connexion ∇ on a Riemannian manifold $(M^n, g)(n > 3)$ such that

$$(D_V D_U R)(X,Y)Z = B(U,V)R(X,Y)Z \qquad (1)$$

Where R is the curvature tensor of ∇ ,B is a (0, 2) tensor and D denotes the Riemannian connexion. It is shown that if (1) is satisfied, i.e., if the curvature tensor $\bar{R}$ of ∇ is 2-recurrent with respect to to the Riemannian connexion, then the manifold $(M^n, g)(n > 3)$ is conformally 2-recurrent.[1]

Corresponding author: J. P. Singh
Phone. +91-08974134152 Fax. +91-0389-9220873
E-mail: Jp_maths@rediffmail.com

PRELIMINARIES

Let $(M^n, g)(n > 3)$ be an n-dimensional Riemannian manifold with Riemannian connexion D. A linear connexion ∇ on $(M^n, g)(n > 3)$ is said to be semi-symmetric if its torsion tensor S satisfies the following

$$S(X,Y) = \eta(Y)X - \eta(X)Y \qquad (1.1)$$

Where η is a 1-form.

We know that if ∇ is a semi-symmetric metric connexion then

$$\nabla_X Y = D_X Y + \eta(Y)X - g(X,Y)\xi \qquad (1.2)$$

Where $g(X,\xi) = \eta(X)$. (1.3)

For every vector field X. Further, it is also known as if K denotes the curvature tensor of D, then

$$R(X,Y)Z = K(X,Y)Z - \alpha(Y,Z)X + \alpha(X,Z)Y - g(Y,Z)\rho X + g(X,Z)\rho Y \qquad (1.4)$$

Where α is a tensor field of type (0, 2) defined by

$$\alpha(X,Y) = (D_X\eta)(Y) - \eta(X)\eta(Y) + \frac{1}{2}\eta(\xi)g(X,Y) \qquad (1.5)$$

and ρ is a tensor field of type (1, 1) defined by

$$g(\rho X, Y) = \alpha(X,Y). \qquad (1.6)$$

For any vector field X, Y.

A SPECIAL TYPE OF SEMI-SYMMETRIC CONNEXION

In this section we consider a semi-symmetric connexion ∇ on (M^n, g) whose curvature tensor satisfies the condition[2]

$$(D_V D_U R)(X,Y)Z = B(U,V)R(X,Y)Z \qquad (2.1)$$

Where B is a (0, 2) type tensor.
From (1.4) we have

$$(D_V D_U R)(X,Y) = (D_V D_U K)(X,Y)Z - [(D_V D_U \alpha)(Y,Z)]X + [(D_V D_U \alpha)(X,Z)]Y - g(Y,Z)(D_V D_U\, \rho)(X) + g(X,Z)(D_V D_U\, \rho)(Y). \qquad (2.2)$$

Using (2.1) in (2.2), we have

$$[(D_V D_U K)(X,Y)Z - B(U,V)K(X,Y)Z] - [(D_V D_U \alpha)(Y,Z) - B(U,V)\alpha(Y,Z)]X + [(D_V D_U \alpha)(X,Z) - B(U,V)\alpha(X,Z)]Y - g(Y,Z)[(D_V D_U \rho)(X) - B(U,V)\rho X] + g(X,Z)[(D_V D_U \rho)(Y) - B(U,V)\rho Y] = 0. \qquad (2.3)$$

The M-Projective curvature tensor is given by[2]

$$W^*(X,Y)Z = K(X,Y)Z - \frac{1}{2(n-1)}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)RX - g(X,Z)RY\}. \qquad (2.4)$$

It can be written as

$$W^*(X,Y)Z = K(X,Y)Z + \lambda(Y,Z)X - \lambda(X,Z)Y + g(Y,Z)MX - g(X,Z)MY \qquad (2.5)$$

Where $\lambda(Y,Z) = -\frac{1}{2(n-1)}S(Y,Z)$ (2.6)

and M is a (1,1) tensor field such that

$$g(MY,Z) = \lambda(Y,Z) \qquad (2.7)$$

Where S is Ricci tensor of (M^n, g) .

DEFINITION

A manifold (M^n, g) (n>3), whose curvature tensor $\bar{R}$ of ∇ is 2-recurrent with respect to the Riemannian connection D and satisfies

$$(D_V D_U W^*)(X,Y)Z = B(U,V)W^*(X,Y,Z)$$

is called M-Projectively 2-recurrent.
From (2.5), we have

$$(D_V D_U W)(X,Y) = (D_V D_U K)(X,Y)Z + [(D_V D_U \lambda)(Y,Z)]X - [(D_V D_U \lambda)(X,Z)]Y + g(Y,Z)(D_V D_U\, M)(X) - g(X,Z)(D_V D_U\, M)(Y). \qquad (2.8)$$

Since Dg=0, we have
$(D_V D_U g)(MY,Z) = 0$, from which we get in virtue of (2.7)

$$(D_V D_U \lambda)(Y,Z) = g[(D_V D_U M)(Y), Z] \qquad (2.9)$$

and by using (1.6), we get

$$(D_V D_U \alpha)(Y,Z) = g[(D_V D_U \rho)(Y), Z]. \qquad (2.10)$$

Now from (2.1), we get

$$(D_V D_U S')(Y,Z) = B(U,V)S'(Y,Z) \qquad (2.11)$$

and $(D_V D_U r') = B(U,V)r'$ (2.12)

Where S' and r' are the Ricci tensor and the scalar curvature of ∇ respectively. Again from (1.4), it follows that

$$S'(Y,Z) = S(Y,Z) - (n-2)\alpha(Y,Z) - \sigma g(Y,Z) \qquad (2.13)$$

and

$$r' = r - 2(n-1)\sigma \qquad (2.14)$$

Where σ is the trace of ρ.

From (2.13), we have

$$(D_V D_U S')(Y,Z) = (D_V D_U S)(Y,Z) - (n-2)(D_V D_U \alpha)(Y,Z) - (D_V D_U \sigma)g(Y,Z). \qquad (2.15)$$

Again from (2.14), we have

$$D_V D_U r' = D_V D_U r - 2(n-1)D_V D_U \sigma.$$

By using (2.12) and (2.14) in above, we obtain

$$D_V D_U \sigma - B(U,V)\sigma = \frac{1}{2(n-1)}[D_V D_U r - B(U,V)r] \qquad (2.16)$$

In view of (2.11), the equation (2.15) can be express as

$$B(U,V)S'(Y,Z) = (D_V D_U S)(Y,Z) - (n-2)(D_V D_U \alpha)(Y,Z) - (D_V D_U \sigma)g(Y,Z). \qquad (2.17)$$

This in virtue of (2.13) and (2.16) gives

$$(n-2)[(D_V D_U \alpha)(Y,Z) - B(U,V)\alpha(Y,Z)] = [(D_V D_U Q)(Y,Z) - B(U,V)Q(Y,Z)] - \frac{1}{2(n-1)} g(Y,Z)[D_V D_U r - B(U,V)r].$$

Now using (2.5) in above, we obtain

$$(D_V D_U \alpha)(Y,Z) - B(U,V)\alpha(Y,Z) = B(U,V)\lambda(Y,Z) - (D_V D_U \lambda)(Y,Z).$$

In virtue of

$$g[g(Y,Z)(D\_V\ D\_U\ \rho)(X), W] = g[-g(Y,Z)(D\_V\ D\_U\ M)(X) + g(Y,Z)B(U,V)MX + g(Y,Z)B(U,V)\rho X, W].$$

From this, it follows that

$$g(Y,Z)(D_V D_U \rho)(X) - g(Y,Z)B(U,V)\rho X = g(Y,Z)B(U,V)MX - g(Y,Z)(D_V D_U M)(X). \qquad (2.18)$$

Using (2.17) and (2.18), the equation (2.3) can be expressed as

$$(D_V D_U K)(X,Y)Z + [(D_V D_U \lambda)(Y,Z)]X - [(D_V D_U \lambda)(X,Z)]Y + g(Y,Z)(D_V D_U M)(X) - g(Y,Z)(D_V D_U M)(Y)$$

$$= B(U,V)[K(X,Y)Z + \lambda(Y,Z)X - \lambda(X,Z)Y \; g(Y,Z)MX - g(X,Z)MY].$$

This gives in virtue of (2.7) and (2.4)

$$(D_V D_U W^*)(X,Y)Z = B(U,V)W^*(X,Y)Z \qquad (2.19)$$

From (2.19) we can state the following theorem.

THEOREM

If a Riemannian manifold $(M^n, g)(n > 3)$ admits a semi-symmetric metric connexion whose curvature tensor is a 2-recurrent tensor with respect to the Riemannian connexion having B as its associated tensor, and then the manifold $(M^n, g)(n > 3)$ is M-Projectively 2-recurrent with B as its tensor of recurrence.

If, in particular B is zero, then the curvature tensor of ∇ becomes a semi-symmetric with respect to Riemannian connexion and consequently $(M^n, g)(n > 3)$ becomes M-Projectively 2-symmetric. Hence we have the following corollary.

COROLLARY

If a Riemannian manifold $(M^n, g)(n > 3)$ admits a semi-symmetric metric connexion whose curvature tensor is 2-symmetric with respect to the Riemannian connexion, then $(M^n, g)(n > 3)$ is M-Projectively 2-symmetric.

ACKNOWLEDGEMENT

The Author is very thankful to Prof. R.H. Ojha for his valuable suggestions.

REFERENCES

1. Thompson AH (1970). *Bull Acad Polon Sci Math Astr Phys*, **18,** 335-340.
2. Uday C (1990). On a type semi-symmetric connection on a Riemannian manifold. *Indian J Pure appl Math*, **21**, 334-338.

